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Third Memoir on a New Theory of Symmetric Functions.

By Major P. A. MacMahon, R. A.

In this memoir I carry on the development of the Theory of Separations. It is divided into seven sections (8 to 14), and is numbered continuously with the second memoir. It brings the theory up to the point where modes of calculating tables of separations may be advantageously discussed. Sections 8 and 9 lay down the fundamental laws of operation which in sections 10, 11 and 12 are applied to the deduction of some comprehensive theorems of algebraic symmetry. Section 13 is concerned with the multiplication of symmetric functions. Section 14 commences the application of the operators to the functions which appear in a table of separations and establishes the theorem which is preliminary to further researches which may possibly appear in a future number of this journal.

I desire to draw attention to the fundamental theorem in operations of which a statement merely is given in Art. 140 of Section 9. It is a generalization of a theorem of Sylvester, to be found in the Philosophical Magazine, 1877, under the title, "A generalization of Taylor's theorem."

§8.

The Differential Operators.

120. I purpose to keep in view throughout the following investigation the analogy between quantity and operation which was pointed out by Hammond for the first time in Vol. XIII of the Proceedings of the London Mathematical Society.

I present the analogy in an extended form and from two distinct points of view.

Supposing n to be indefinitely great, I write down two relations, viz:

$$1 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n$$
$$= (1 + \alpha_1x)(1 + \alpha_2x)(1 + \alpha_3x)\dots(1 + \alpha_nx),$$
$$1 + a_{-1}\frac{1}{x} + a_{-2}\frac{1}{x^2} + a_{-3}\frac{1}{x^3} + \dots + a_{-n}\frac{1}{x^n}$$
$$= \left(1 + \frac{1}{\alpha_1x}\right)\left(1 + \frac{1}{\alpha_2x}\right)\left(1 + \frac{1}{\alpha_3x}\right)\dots\left(1 + \frac{1}{\alpha_nx}\right),$$

which lead to the algebraic theory of the expression of any rational function of the quantities

$$\alpha_1,\ \alpha_2,\ \alpha_3,\ \dots\ \alpha_n$$

as a rational *integral* function of the number n and of the quantities

$$a_1,\ a_{-1},\ a_2,\ a_{-2},\ a_3,\ a_{-3},\ \dots\ a_n,\ a_{-n}.$$

121. By multiplying the two equations together and subsequently multiplying each side by exp n, we obtain

$$e^n\left(1 + a_1x + a_2x^2 + a_3x^3 + \dots\right)\left(1 + a_{-1}\frac{1}{x} + a_{-2}\frac{1}{x^2} + a_{-3}\frac{1}{x^3} + \dots\right)$$
$$= e^n\left(1 + \alpha_1x\right)\left(1 + \alpha_2x\right)\left(1 + \alpha_3x\right)\dots\left(1 + \frac{1}{\alpha_1x}\right)\left(1 + \frac{1}{\alpha_2x}\right)\left(1 + \frac{1}{\alpha_3x}\right)\dots$$

Observe that the right-hand side is

$$e^n\{1 + (1)x + (1^2)x^2 + (1^3)x^3 + \dots\}\left\{1 + (\bar{1})\frac{1}{x} + (\bar{1}^2)\frac{1}{x^2} + (\bar{1}^3)\frac{1}{x^3} + \dots\right\},$$

that is,

$$e^n\Big[1 + (1)x + (\bar{1})\frac{1}{x} + (1^2)x^2 + (1)(\bar{1}) + (\bar{1}^2)\frac{1}{x^2} + (1^3)x^3 + (1^2)(\bar{1})x$$
$$+ (1)(\bar{1}^2)\frac{1}{x} + (\bar{1}^3)\frac{1}{x^3} + \dots\Big],$$

and this is

$$e^n\Big\{1 + (1)x + (\bar{1})\frac{1}{x} + (1^2)x^2 + (0) + (1\bar{1}) + (\bar{1}^2)\frac{1}{x^2}$$
$$+ (1^3)x^3 + (10)x + (1^2\bar{1})x + (0\bar{1})\frac{1}{x} + (1\bar{1}^2)\frac{1}{x} + (\bar{1}^3)\frac{1}{x^3} + \dots\Big\}$$
$$= e^n\sum(1^\kappa 0^\lambda \bar{1}^\mu)\,x^{\kappa-\mu},$$

the summation being for all zero and positive integer values of κ, λ and μ.

But
$$\sum(1^\kappa 0^\lambda \bar{1}^\mu)\,x^{\kappa-\mu} = \exp\left\{(0) + (1)x + (\bar{1})\frac{1}{x}\right\}$$

symbolically, where

$$\frac{(1)^{\kappa}(0)^{\lambda}(\bar{1})^{\mu}}{\kappa!\ \lambda!\ \mu!} \text{ is the symbolic expression for } (1^{\kappa}0^{\lambda}\bar{1}^{\mu}).$$

Hence we may write

$$e^{n}(1 + a_1x + a_2x^2 + \ldots.)\left(1 + a_{-1}\frac{1}{x} + a_{-2}\frac{1}{x^2} + \ldots.\right)$$
$$= e^{n} \exp\left\{(0) + (1)x + (\bar{1})\frac{1}{x}\right\} \text{ symbolically.}$$

Returning to the former identity and taking logarithms, we find

$$n + \log(1 + a_1x + a_2x^2 + \ldots.) + \log\left(1 + a_{-1}\frac{1}{x} + a_{-2}\frac{1}{x^2} + \ldots.\right)$$
$$= (0) + (1)x - \tfrac{1}{2}(2)x^2 + \tfrac{1}{3}(3)x^3 - \ldots.$$
$$+ (\bar{1})\frac{1}{x} - \tfrac{1}{2}(\bar{2})\frac{1}{x^2} + \tfrac{1}{3}(\bar{3})\frac{1}{x^3} - \ldots.,$$

an identity which indicates that any symmetric function which can be exhibited by means of partitions composed of positive, zero and negative integers, is uniquely expressible by the use of the quantities

$$n,\ a_1,\ a_{-1},\ a_2,\ a_{-2},\ \ldots.;$$

it further gives the law of such expression of the sums of the powers of the quantities

$$\alpha_1,\ \alpha_2,\ \alpha_3,\ \ldots.$$

122. We are now led to the result

$$e^{n} \exp\left\{(0) + (1)x + (\bar{1})\frac{1}{x}\right\} \text{ symbolic}$$
$$= \exp\left\{\begin{array}{l}(0) + (1)x - \tfrac{1}{2}(2)x^2 + \tfrac{1}{3}(3)x^3 - \ldots. \\ \quad + (\bar{1})\frac{1}{x} - \tfrac{1}{2}(\bar{2})\frac{1}{x^2} + \tfrac{1}{3}(\bar{3})\frac{1}{x^3} - \ldots.,\end{array}\right\}$$

which gives

$$\exp\left\{(0) + (1)x + (\bar{1})\frac{1}{x}\right\} \text{ symbolic}$$
$$= \exp\left\{\begin{array}{l}\quad (1)x - \tfrac{1}{2}(2)x^2 + \tfrac{1}{3}(3)x^3 - \ldots. \\ + (\bar{1})\frac{1}{x} - \tfrac{1}{2}(\bar{2})\frac{1}{x^2} + \tfrac{1}{3}(\bar{3})\frac{1}{x^3} - \ldots.\end{array}\right\}$$

a formula of great importance, as will appear presently.

123. Write now

$$
\begin{aligned}
&1 + A_0 + A_1 x + A_2 x^2 + \dots\dots \\
&\qquad + A_{-1}\frac{1}{x} + A_{-2}\frac{1}{x^2} + \dots\dots \\
&= \exp\left\{\begin{aligned}&(0) + (1)x - \tfrac{1}{2}(2)x^2 + \tfrac{1}{3}(3)x^3 - \dots\dots \\ &+ (\bar{1})\frac{1}{x} - \tfrac{1}{2}(\bar{2})\frac{1}{x^2} + \tfrac{1}{3}(\bar{3})\frac{1}{x^3} - \dots\dots\end{aligned}\right\}
\end{aligned}
$$

so that

$$
\begin{aligned}
1 + A_0 &= e^n(1 + a_1 a_{-1} + a_2 a_{-2} + a_3 a_{-3} + \dots\dots), \\
A_1 &= e^n(a_1 + a_2 a_{-1} + a_3 a_{-2} + \dots\dots), \\
A_{-1} &= e^n(a_{-1} + a_1 a_{-2} + a_2 a_{-3} + \dots\dots), \\
A_2 &= e^n(a_2 + a_3 a_{-1} + a_4 a_{-2} + \dots\dots), \\
A_{-2} &= e^n(a_{-2} + a_1 a_{-3} + a_2 a_{-4} + \dots\dots), \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
$$

124. The quantities

$$A_0,\ A_1,\ A_{-1},\ A_2,\ A_{-2},\ \dots\dots,$$

now, I believe, introduced into analysis for the first time, are of great and fundamental importance in the theory; subsequently they will be freely adopted as arguments of the symmetric functions, and it will appear that, for the purposes of analysis, they are the proper functions to consider.

125. As I wish to show the complete correspondence which exists between the algebraic theory and the theory of the related partial differential operations, it is necessary to write down some obvious results which, however, are in correspondence with theorems in operations which are by no means obvious.

Thus the last written identity, through expansion of the exponential function and comparison of coefficients, leads to the manifest conclusions:

$$
\begin{aligned}
1 + A_0 &= e^{(0)}\left\{1 + (1)(\bar{1}) + \frac{(1)^2 - (2)}{2!}\cdot\frac{(\bar{1})^2 - (\bar{2})}{2!}\right. \\
&\qquad\left. + \frac{(1)^3 - 3(2)(1) + 2(3)}{3!}\cdot\frac{(\bar{1})^3 - 3(\bar{2})(\bar{1}) + 2(\bar{3})}{3!} + \dots\dots\right\}, \\
A_1 &= e^{(0)}\left\{(1) + \frac{(1)^2 - (2)}{2!}(\bar{1}) + \frac{(1)^3 - 3(2)(1) + 2(3)}{3!}\cdot\frac{(\bar{1})^2 - (\bar{2})}{2!} + \dots\dots\right\}, \\
A_{-1} &= e^{(0)}\left\{(\bar{1}) + \frac{(\bar{1})^2 - (\bar{2})}{2!}(1) + \frac{(\bar{1})^3 - 3(\bar{2})(\bar{1}) + 2(\bar{3})}{3!}\cdot\frac{(1)^2 - (2)}{2!} + \dots\dots\right\}, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
$$

relations which also are simply reached from the expressions of A_0, A_1, A_{-1}, ... in terms of n, a_1, a_{-1}, a_2, a_{-2} .... by expressing these last quantities in terms of the sums of the powers.

126. Again, from the identity,

$$\begin{aligned}
&1 + A_0 + A_1x \quad + A_2x^2 \quad + A_3x^3 \quad + \ldots\ldots \\
&\qquad + A_{-1}\frac{1}{x} + A_{-2}\frac{1}{x^2} + A_{-3}\frac{1}{x^3} + \ldots\ldots \\
&= \exp\left\{\begin{aligned}&(0) + (1)x \quad - \tfrac{1}{2}(2)x^2 \quad + (3)\tfrac{1}{3}x^3 \quad - \ldots\ldots \\ &+ (\bar{1})\frac{1}{x} - \tfrac{1}{2}(\bar{2})\frac{1}{x^2} + \tfrac{1}{3}(\bar{3})\frac{1}{x^3} - \ldots\ldots\end{aligned}\right\}
\end{aligned}$$

we obtain by taking logarithms,

$$\begin{aligned}
&(0) + (1)x \quad - \tfrac{1}{2}(2)x^2 \quad + \tfrac{1}{3}(3)x^3 \quad - \ldots\ldots \\
&\qquad + (\bar{1})\frac{1}{x} - \tfrac{1}{2}(\bar{2})\frac{1}{x^2} + \tfrac{1}{3}(\bar{3})\frac{1}{x^3} - \ldots\ldots \\
&= \log\left(\begin{aligned}&1 + A_0 + A_1x \quad + A_2x^2 \quad + A_3x^3 \quad + \ldots\ldots \\ &+ A_{-1}\frac{1}{x} + A_{-2}\frac{1}{x^2} + A_{-3}\frac{1}{x^3} + \ldots\ldots\end{aligned}\right)
\end{aligned}$$

and on expanding and comparing coefficients of like powers of x, we have

$$\begin{aligned}
(0) = \log(1 + A_0) - \frac{A_1A_{-1}}{(1+A_0)^2} \qquad & - \frac{A_2A_{-2}}{(1+A_0)^2} \qquad && + \frac{A_1^2A_{-2}}{(1+A_0)^3} \\
& - \frac{A_3A_{-3}}{(1+A_0)^2} && + 2\frac{A_2A_1A_{-3}}{(1+A_0)^3} \\
- \frac{A_1^3A_{-3}}{(1+A_0)^4} - \ldots\ldots & + \frac{A_2A_{-1}^2}{(1+A_0)^3} && - \frac{3}{2}\,\frac{A_1^2A_{-1}^2}{(1+A_0)^4} \\
+ 2\frac{A_3A_{-1}A_{-2}}{(1+A_0)^3} \qquad & - 6\frac{A_2A_1A_{-1}A_{-2}}{(1+A_0)^4} && + 4\frac{A_1^3A_{-1}A_{-2}}{(1+A_0)^5} + \ldots\ldots \\
- \frac{A_3A_{-1}^3}{(1+A_0)^4} \qquad & + 4\frac{A_2A_1A_{-1}^3}{(1+A_0)^5} && - \frac{10}{3}\,\frac{A_1^3A_{-1}^3}{(1+A_0)^6} - \ldots\ldots
\end{aligned}$$

$$= \sum \frac{(-)^{\Sigma\pi-1}(\Sigma\pi - 1)!}{\ldots\pi_3!\,\pi_2!\,\pi_1!\,\pi_0!\,\pi_{-1}!\,\pi_{-2}!\,\pi_{-3}!\ldots} \ldots A_3^{\pi_3}A_2^{\pi_2}A_1^{\pi_1}A_0^{\pi_0}A_{-1}^{\pi_{-1}}A_{-2}^{\pi_{-2}}A_{-3}^{\pi_{-3}}\ldots,$$

the summation being in regard to every solution of the indeterminate equation

$$\Sigma t\pi_t = 0$$

in positive, zero and negative integers.

127. In general,

$$\frac{(-)^{m-1}}{m}(m) = \sum \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\ldots \pi_2!\,\pi_1!\,\pi_0!\,\pi_{-1}!\,\pi_{-2}!\ldots} \ldots A_2^{\pi_2} A_1^{\pi_1} A_0^{\pi_0} A_{-1}^{\pi_{-1}} A_{-2}^{\pi_{-2}} \ldots,$$

$$\frac{(-)^{m-1}}{m}(\overline{m}) = \sum \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\ldots \pi_2!\,\pi_1!\,\pi_0!\,\pi_{-1}!\,\pi_{-2}!\ldots} \ldots A_2^{\pi_2} A_1^{\pi_1} A_0^{\pi_0} A_{-1}^{\pi_{-1}} A_{-2}^{\pi_{-2}} \ldots,$$

the summations being in regard to the solutions of the indeterminate equations

$$\Sigma t\pi_t = m,$$

$$\Sigma t\pi_t = -m$$

respectively, in positive, zero and negative integers.

§9.

128. We may express symmetric functions, of the nature here considered, as functions either of the quantities

$$n,\ a_1,\ a_{-1},\ a_2,\ a_{-2},\ \ldots.$$

or of the quantities

$$A_0,\ A_1,\ A_{-1},\ A_2,\ A_{-2},\ \ldots.,$$

where observe that the latter set of arguments does not involve the number n explicitly. Both sets of quantities are defined by the identities

$$\begin{aligned} & e^n(1 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots.)\left(1 + a_{-1}\frac{1}{x} + a_{-2}\frac{1}{x^2} + a_{-3}\frac{1}{x^3} + \ldots.\right) \\ = {} & 1 + A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \ldots. \\ & \qquad + A_{-1}\frac{1}{x} + A_{-2}\frac{1}{x^2} + A_{-3}\frac{1}{x^3} + \ldots. \\ = {} & e^n(1+\alpha_1 x)(1+\alpha_2 x)(1+\alpha_3 x)\ldots.\left(1+\frac{1}{\alpha_1 x}\right)\left(1+\frac{1}{\alpha_2 x}\right)\left(1+\frac{1}{\alpha_3 x}\right)\ldots. \end{aligned}$$

129. Let us now introduce a quantity μ in addition to the n quantities

$$\alpha_1,\ \alpha_2,\ \alpha_3,\ \ldots.\ \alpha_n.$$

We thus obtain the identities

$$\begin{aligned} & e^{n+1}(1+a_1 x+a_2 x^2+a_3 x^3+\ldots)\left(1+a_{-1}\frac{1}{x}+a_{-2}\frac{1}{x^2}+a_{-3}\frac{1}{x^3}+\ldots\right)(1+\mu x)\left(1+\frac{1}{\mu x}\right) \\ = {} & \left(\begin{array}{l} 1 + A_0 + A_1 x + A_2 x^2 + \ldots. \\ \qquad + A_{-1}\frac{1}{x} + A_{-2}\frac{1}{x^2} + \ldots. \end{array}\right) e(1+\mu x)\left(1+\frac{1}{\mu x}\right) \\ = {} & e^{n+1}(1+\alpha_1 x)(1+\alpha_2 x)\ldots.(1+\mu x)\left(1+\frac{1}{\alpha_1 x}\right)\left(1+\frac{1}{\alpha_2 x}\right)\ldots.\left(1+\frac{1}{\mu x}\right), \end{aligned}$$

for it is merely necessary to multiply throughout by the factor

$$e(1+\mu x)\left(1+\frac{1}{\mu x}\right).$$

130. The identities may be written

$$\begin{aligned}
&e^{n+1}\{1+(a_1+\mu)x+(a_2+a_1\mu)x^2+(a_3+a_2\mu)x^3+\ldots\}\\
&\quad\times\left\{1+\left(a_{-1}+\frac{1}{\mu}\right)\frac{1}{x}+\left(a_{-2}+a_{-1}\frac{1}{\mu}\right)\frac{1}{x^2}+\left(a_{-3}+a_{-2}\frac{1}{\mu}\right)\frac{1}{x^3}+\ldots\right\}\\
&=1+A_0\ +(2e-1)(1+A_0)+eA_{-1}\mu+eA_1\frac{1}{\mu}\\
&\quad+A_1\ +(2e-1)A_1+e(1+A_0)\mu\ +eA_2\frac{1}{\mu}\\
&\quad+A_{-1}+(2e-1)A_{-1}+eA_{-2}\mu+e(1+A_0)\frac{1}{\mu}\\
&\quad+A_2\ +(2e-1)A_2+eA_1\mu+eA_3\frac{1}{\mu}\\
&\quad+A_{-2}+(2e-1)A_{-2}+eA_{-3}\mu+eA_{-1}\frac{1}{\mu}\\
&\quad+\ldots\ldots\ldots\ldots\ldots\ldots\\
&=e^{n+1}(1+\alpha_1x)(1+\alpha_2x)\ldots(1+\mu x)\left(1+\frac{1}{\alpha_1x}\right)\left(1+\frac{1}{\alpha_2x}\right)\ldots\left(1+\frac{1}{\mu x}\right),
\end{aligned}$$

from which it appears that the introduction of the new quantity μ has the effect of changing the quantities

$$\ldots a_{-3}\ ,\quad a_{-2}\ ,\quad a_{-1}\ ,\quad n\ ,\quad a_1\ ,\quad a_2\ ,\quad a_3\ ,\ \ldots$$

into

$$a_{-3}+a_{-2}\frac{1}{\mu},\ a_{-2}+a_{-1}\frac{1}{\mu},\ a_{-1}+\frac{1}{\mu},\ n+1,\ a_1+\mu,\ a_2+a_1\mu,\ a_3+a_2\mu,\ \ldots$$

respectively, and moreover changes

$$\begin{aligned}
&A_0\ \text{into}\ A_0\ +(2e-1)(1+A_0)+eA_{-1}\mu+eA_1\frac{1}{\mu},\\
&A_1\ \text{into}\ A_1\ +(2e-1)A_1+e(1+A_0)\mu+eA_2\frac{1}{\mu},\\
&A_{-1}\ \text{into}\ A_{-1}+(2e-1)A_{-1}+eA_{-2}\mu+e(1+A_0)\frac{1}{\mu},\\
&A_2\ \text{into}\ A_2\ +(2e-1)A_2+eA_1\mu+eA_3\frac{1}{\mu},\\
&A_{-2}\ \text{into}\ A_{-2}+(2e-1)A_{-2}+eA_{-3}\mu+eA_{-1}\frac{1}{\mu},\\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

131. Hence if any symmetric function be

$$\begin{aligned}\phi\{n,\ a_1,\ a_{-1},\ a_2,\ a_{-2},\ a_3,\ a_{-3},\ \ldots.\} &= \psi\{A_0,\ A_1,\ A_{-1},\ A_2,\ A_{-2},\ \ldots.\}\\ &= \phi = \psi,\end{aligned}$$

the altered value will be

$$\phi\left\{n+1,\ a_1+\mu,\ a_{-1}+\frac{1}{\mu},\ a_2+a_1\mu,\ a_{-2}+a_{-1}\frac{1}{\mu},\ \ldots.\right\},$$

which by Taylor's theorem is symbolically

$$\exp\left(d_0+d_1\mu+d_{-1}\frac{1}{\mu}\right)\phi,$$

where

$$\begin{aligned} d_0 &= \partial_n,\\ d_1 &= \partial_{a_1} + a_1\partial_{a_2} + a_2\partial_{a_3} + \ldots.,\\ d_{-1} &= \partial_{a_{-1}} + a_{-1}\partial_{a_{-2}} + a_{-2}\partial_{a_{-3}} + \ldots.,\end{aligned}$$

and the multiplication of operators is symbolic, and will also be

$$\begin{aligned}\psi\Big\{&A_0 + (2e-1)(1+A_0) + eA_{-1}\mu + eA_1\frac{1}{\mu},\\ &A_1 + (2e-1)A_1 + e(1+A_0)\mu + eA_2\frac{1}{\mu},\\ &A_{-1} + (2e-1)A_{-1} + eA_{-2}\mu + e(1+A_0)\frac{1}{\mu},\ \ldots.\Big\}\end{aligned}$$

which by Taylor's theorem is symbolically

$$\exp\left\{e\left(g_0+g_1\mu+g_{-1}\frac{1}{\mu}\right)+(e-1)\,g_0\right\}.\psi,$$

where

$$\begin{aligned} g_0 &= (1+A_0)\,\partial_{A_0} + A_1\partial_{A_1} + A_2\partial_{A_2} + \ldots.\\ &\qquad + A_{-1}\partial_{A_{-1}} + A_{-2}\partial_{A_{-2}} + \ldots.,\\ g_1 &= (1+A_0)\,\partial_{A_1} + A_1\partial_{A_2} + A_2\partial_{A_3} + \ldots.\\ &\qquad + A_{-1}\partial_{A_0} + A_{-2}\partial_{A_{-1}} + \ldots.,\\ g_{-1} &= (1+A_0)\,\partial_{A_{-1}} + A_1\partial_{A_0} + A_2\partial_{A_1} + \ldots.\\ &\qquad + A_{-1}\partial_{A_{-2}} + A_{-2}\partial_{A_{-3}} + \ldots.,\end{aligned}$$

and the multiplication of operators is symbolic.

132. If, moreover, the symmetric function be

$$(pq\ \ldots.\ o^{\omega}\bar{r}\bar{s}\ \ldots.) = \phi = \psi,$$

the introduction of the new quantity μ results in the addition to $(pq\ \ldots\ o^{\omega}\bar{r}\bar{s}\ \ldots)$ of the new terms

$$\begin{aligned}&(q\ \ldots.\ o^{\omega}\bar{r}\bar{s}\ \ldots.)\,\mu^{p} + (p\ \ldots.\ o^{\omega}\bar{r}\bar{s}\ \ldots.)\,\mu^{q} + \ldots.\\ &+ (pq\ \ldots.\ o^{\omega-1}\bar{r}\bar{s})\,\mu^{o} + (pq\ \ldots.\ o^{\omega}\bar{s}\ \ldots.)\frac{1}{\mu^{r}} + (pq\ \ldots.\ o^{\omega}\bar{r}\ \ldots.)\frac{1}{\mu^{s}} + \ldots.,\end{aligned}$$

so that we are face to face with the identities

$$\begin{aligned}(pq \ldots\ldots o^{\omega}\bar{r}\bar{s} \ldots\ldots) &+ (pq \ldots\ldots o^{\omega-1}\bar{r}\bar{s})\,\mu^{o} \\ &+ (q \ldots o^{\omega}\bar{r}\bar{s} \ldots\ldots)\,\mu^{p} \quad + (p \ldots\ldots o^{\omega}\bar{r}\bar{s} \ldots\ldots)\,\mu^{q} + \ldots\ldots \\ &+ (pq \ldots\ldots o^{\omega}\bar{s} \ldots\ldots)\frac{1}{\mu^{r}} + (pq \ldots\ldots o^{\omega}\bar{r} \ldots\ldots)\frac{1}{\mu^{s}} + \ldots\ldots \\ &= \exp\left(d_0 + d_1\mu + d_{-1}\frac{1}{\mu}\right).\phi \\ &= \exp\left\{e\left(g_0 + g_1\mu + g_{-1}\frac{1}{\mu}\right) + (e-1)\,g_0\right\}.\psi.\end{aligned}$$

133. We now write

$$\begin{aligned}\exp\left(d_0 + d_1\mu + d_{-1}\frac{1}{\mu}\right) &= 1 + D_0\mu^0 + D_1\mu \quad + D_2\mu^2 \quad + \ldots \\ &\qquad + D_{-1}\frac{1}{\mu} + D_{-2}\frac{1}{\mu^2} + \ldots, \\ \exp\left\{e\left(g_0 + g_1\mu + g_{-1}\frac{1}{\mu}\right) + (e-1)\,g_0\right\} &= 1 + G_0\mu^0 + G_1\mu \quad + G_2\mu^2 \quad + \ldots \\ &\qquad + G_{-1}\frac{1}{\mu} + G_{-2}\frac{1}{\mu^2} + \ldots,\end{aligned}$$

the expansions being of course symbolic, and we find on equating coefficients of like powers of μ,

$$\begin{aligned}D_0\ (pq \ldots\ldots o^{\omega}\bar{r}\bar{s} \ldots\ldots) &= G_o\ (pq \ldots\ldots o^{\omega}\bar{r}\bar{s} \ldots\ldots) = (pq \ldots\ldots o^{\omega-1}\bar{r}\bar{s} \ldots\ldots), \\ D_p\ (pq \ldots\ldots o^{\omega}\bar{r}\bar{s} \ldots\ldots) &= G_p\ (pq \ldots\ldots o^{\omega}\bar{r}\bar{s} \ldots\ldots) = (q \ \ldots\ldots o^{\omega}\ \ \bar{r}\bar{s} \ldots\ldots), \\ D_q\ (pq \ldots\ldots o^{\omega}\bar{r}\bar{s} \ldots\ldots) &= G_q\ (pq \ldots\ldots o^{\omega}\bar{r}\bar{s} \ldots\ldots) = (p \ \ldots\ldots o^{\omega}\ \ \bar{r}\bar{s} \ldots\ldots), \\ D_{-r}(pq \ldots\ldots o^{\omega}\bar{r}\bar{s} \ldots\ldots) &= G_{-r}(pq \ldots\ldots o^{\omega}\bar{r}\bar{s} \ldots\ldots) = (pq \ldots\ldots o^{\omega}\ \ \bar{s} \ldots\ldots), \\ D_{-s}(pq \ldots\ldots o^{\omega}\bar{r}\bar{s} \ldots\ldots) &= G_{-s}(pq \ldots\ldots o^{\omega}\bar{r}\bar{s} \ldots\ldots) = (pq \ldots\ldots o^{\omega}\ \ \bar{r} \ldots\ldots), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

or x, being positive, zero, or negative, D_x or G_x is an operating symbol which, performed upon a monomial symmetric function, has the effect of striking out one part x from its partition.

Also, if P denote a partition which contains no part x,

$$D_xP = G_xP = \text{zero}.$$

Moreover, $$D_x(x) = G_x(x) = 1.$$

(Compare Hammond, Proceedings of the London Mathematical Society, Vol. XIII, p. 79.)

134. As remarked above, in the two results

$$1 + D_0 + D_1\mu + D_2\mu^2 + \ldots . = \exp\left(d_0 + d_1\mu + d_{-1}\frac{1}{\mu}\right)$$
$$+ D_{-1}\frac{1}{\mu} + D_{-2}\frac{1}{\mu^2} + \ldots . ,$$
$$1 + G_0 + G_1\mu + G_2\mu^2 + \ldots . = \exp\left\{e\left(g_0 + g_1\mu + g_{-1}\frac{1}{\mu}\right) + (e-1)\,g_0\right\}$$
$$+ G_{-1}\frac{1}{\mu} + G_{-2}\frac{1}{\mu^2} + \ldots .$$

the multiplications arising from the developments of the exponential functions are symbolic.

135. It is convenient to indicate this, as Hammond has done, by placing a horizontal line over the operators which are so multiplied.*

136. We have now the relations which follow, obtained at once by a comparison of coefficients on either side of the two identities

$$1 + D_0 = e^{d_0}\left\{1 + \frac{\overline{d_1 d_{-1}}}{1!\,1!} + \frac{\overline{d_1^2 d_{-1}^2}}{2!\,2!} + \frac{\overline{d_1^3 d_{-1}^3}}{3!\,3!} + \ldots .\right\},$$
$$= e^{d_0}\sum_{s=0}^{s=\infty}\frac{\overline{d_1^s d_{-1}^s}}{s!\;s!};$$
$$D_1 = e^{d_0}\left\{d_1 + \frac{\overline{d_1^2 d_{-1}}}{2!\,1!} + \frac{\overline{d_1^3 d_{-1}^2}}{3!\,2!} + \frac{\overline{d_1^4 d_{-1}^3}}{4!\,3!} + \ldots .\right\},$$
$$= e^{d_0}\sum_{s=0}^{s=\infty}\frac{\overline{d_1^{s+1} d_{-1}^s}}{(s+1)!\;s!};$$
$$D_{-1} = e^{d_0}\left\{d_{-1} + \frac{\overline{d_1 d_{-1}^2}}{1!\,2!} + \frac{\overline{d_1^2 d_{-1}^3}}{2!\,3!} + \frac{\overline{d_1^3 d_{-1}^4}}{3!\,4!} + \ldots .\right\},$$
$$= e^{d_0}\sum_{s=0}^{s=\infty}\frac{\overline{d_1^s d_{-1}^{s+1}}}{s!\;(s+1)!};$$

---

* It will be convenient also in future to write $\overline{\exp}\,u$ when the multiplication of operators that occur in u is symbolic, and $\exp u$ in other cases.

and in general

$$D_\kappa \;= e^{d_0}\sum_{s=0}^{s=\infty}\frac{\overline{d_1^{s+\kappa}d_{-1}^{s}}}{(s+\varkappa)!\;s!};$$

$$D_{-\kappa}= e^{d_0}\sum_{s=0}^{s=\infty}\frac{\overline{d_1^{s}d_{-1}^{s+\kappa}}}{s!\;(s+\varkappa)!};$$

whatever be the value of $\varkappa$.

137. Also

$$1+G_0 \;= e^{(2e-1)g_0}\sum_{s=0}^{s=\infty} e^{2s}\,\frac{\overline{g_1^{s}g_{-1}^{s}}}{s!\;s!},$$

$$G_\kappa \;= e^{(2e-1)g_0}\sum_{s=0}^{s=\infty} e^{2s+\kappa}\,\frac{\overline{g_1^{s+\kappa}g_{-1}^{s}}}{(s+\varkappa)!\;s!},$$

$$G_{-\kappa}= e^{(2e-1)g_0}\sum_{s=0}^{s=\infty} e^{2s+\kappa}\,\frac{\overline{g_1^{s}g_{-1}^{s+\kappa}}}{s!\;(s+\varkappa)!}.$$

138. In these relations the factors $\overline{\exp}\,d_0$, $\overline{\exp}\,(2e-1)\,g_0$ are to be multiplied symbolically into all that follows; this is of no importance in the case of the factor $\overline{\exp}\,d_0$, because neither of the operations d_1, d_{-1} involves the quantity n as a symbol of quantity; we may in this case, if we choose, perform the operations

$$\left\{1+\frac{\overline{d_1 d_{-1}}}{1!\;1!}+\dots\right\}$$

and

$$\exp d_0 = 1+\frac{\partial_n}{1!}+\frac{\partial_n^2}{2!}+\dots$$

successively, for their successive operation is equivalent to their symbolic multiplication. Also the symbols which occur in d_1 and d_{-1} are altogether independent, so that the operation

$$\overline{d_1^{s+\kappa}d_{-1}^{s}}$$

may be regarded as the successive performance of the operations

$$\overline{d_1^{s+\kappa}}$$

and

$$\overline{d_{-1}^{s}},$$

and may likewise be written

$$\overline{d_1^{s+\kappa}}\;\overline{d_{-1}^{s}};$$

hence we may also write

$$D_\kappa \;=\sum_{t=0}^{t=\infty}\sum_{s=0}^{s=\infty}\frac{\overline{d_1^{s+\kappa}}\,\overline{d_0^t}\,\overline{d_{-1}^s}}{(s+\varkappa)!\;t!\;s!},$$

$$D_{-\kappa}=\sum_{t=0}^{t=\infty}\sum_{s=0}^{s=\infty}\frac{\overline{d_1^s}\,\overline{d_0^t}\,\overline{d_{-1}^{s+\kappa}}}{s!\;t!\;(s+\varkappa)!}.$$

With regard to the operations g_0, g_1, g_{-1} the case is different, since the same quantities occur as symbols of quantity in all three. Hence it is absolutely necessary to actually perform all the symbolic multiplications.

139. It is now necessary to find the expressions for

$$\overline{\exp}\left(d_0+d_1\mu+d_{-1}\frac{1}{\mu}\right)$$

and

$$\overline{\exp}\left\{e\left(g_0+g_1\mu+g_{-1}\frac{1}{\mu}\right)+(e-1)\,g_0\right\}$$

in series of products of linear operators in which the multiplication is not symbolic.

In the first place write

$$d_\lambda \;=\partial_{a_\lambda}\;+a_1\partial_{a_{\lambda+1}}\;\;+a_2\partial_{a_{\lambda+2}}\;\;+\ldots.$$

$$d_{-\lambda}=\partial_{a_{-\lambda}}+a_{-1}\partial_{a_{-\lambda-1}}+a_{-2}\partial_{a_{-\lambda-2}}+\ldots.$$

Now

$$\overline{\exp}\,d_1\mu=1+d_1\mu+\frac{\overline{d_1^2}}{2!}\mu^2+\frac{\overline{d_1^3}}{3!}\mu^3+\ldots.,$$

and the correspondence between the algebra of the operators d_λ and that of symmetric functions was pointed out and established by Hammond (loc. cit. Proc. Lond. Math. Soc.).

Hence since, as is well known,

$$1+(1)\mu+(1^2)\mu^2+(1^3)\mu^3+\ldots.=\exp\left\{(1)\mu-\tfrac{1}{2}(2)\mu^2+\tfrac{1}{3}(3)\mu^3-\ldots.\right\}$$

we derived at once

$$1+d_1\mu+\frac{\overline{d_1^2}}{2!}\mu^2+\frac{\overline{d_1^3}}{3!}\mu^3+\ldots.=\exp\left(d_1\mu-\tfrac{1}{2}d_2\mu^2+\tfrac{1}{3}d_3\mu^3-\ldots.\right);$$

that is,

$$\overline{\exp}\,d_1\mu=\exp\left(d_1\mu-\tfrac{1}{2}d_2\mu^2+\tfrac{1}{3}d_3\mu^3-\ldots.\right),$$

where the multiplications on the dexter are not symbolic.

Similarly

$$\overline{\exp}\, d_{-1}\frac{1}{\mu} = \exp\left(d_{-1}\frac{1}{\mu} - \tfrac{1}{2} d_{-2}\frac{1}{\mu^2} + \tfrac{1}{3} d_{-3}\frac{1}{\mu^3} - \dots\right).$$

Hence, remarking what has gone before, the absolute identity

$$\overline{\exp}\left(d_0 + d_1\mu + d_{-1}\frac{1}{\mu}\right) = \exp\left\{\begin{aligned} d_0 + d_1\mu \quad &- \tfrac{1}{2} d_2\mu^2 + \tfrac{1}{3} d_3\mu^3 - \dots \\ + d_{-1}\frac{1}{\mu} &- \tfrac{1}{2} d_{-2}\frac{1}{\mu^2} + \tfrac{1}{3} d_{-3}\frac{1}{\mu^3} - \dots \end{aligned}\right\}.$$

140. In the second place, writing in general

$$\begin{aligned} g_\lambda = (1 + A_0)\,\partial_{A_\lambda} + A_1\partial_{A_{\lambda+1}} &+ A_2\partial_{A_{\lambda+2}} + \dots \\ + A_{-1}\partial_{A_{\lambda-1}} &+ A_{-2}\partial_{A_{\lambda-2}} + \dots, \end{aligned}$$

where λ is any positive, zero, or negative integer, it will be convenient to at once put on record a theorem of great generality and importance that I have been led to by the present investigation.

Let

$$\exp\left\{\begin{aligned} f_0 + f_1 y \quad &+ f_2 y^2 + \dots \\ + f_{-1}\frac{1}{y} &+ f_{-2}\frac{1}{y^2} + \dots \end{aligned}\right\} = 1 + \begin{aligned} F_0 + F_1 y \quad &+ F_2 y^2 + \dots \\ + F_{-1}\frac{1}{y} &+ F_{-2}\frac{1}{y^2} + \dots, \end{aligned}$$

where y is arbitrary, be an absolute identity. Then the theorem asserts the absolute identity

$$\exp\left(\begin{aligned} f_0g_0 + f_1g_1 &+ f_2g_2 + \dots \\ + f_{-1}g_{-1} &+ f_{-2}g_{-2} + \dots \end{aligned}\right) = \overline{\exp}\left(\begin{aligned} F_0g_0 + F_1g_1 &+ F_2g_2 + \dots \\ + F_{-1}g_{-1} &+ F_{-2}g_{-2} + \dots \end{aligned}\right).$$

Of this theorem I have two independent proofs which will be communicated elsewhere; it enables us from any linear function P of the operators to determine another linear function Q such that

$$\exp P = \overline{\exp}\, Q,$$

or conversely.

141. In the case before us we are given

$$\overline{\exp}\left\{(2e - 1)\,g_0 + e\mu g_1 + \frac{e}{\mu}\, g_{-1}\right\},$$

so that $F_0 = 2e - 1$, $F_1 = e\mu$, $F_{-1} = \dfrac{e}{\mu}$ and the remainder of the F's vanish.

Hence taking logarithms of the ruling identity

$$\begin{aligned} &f_0 + f_1 y + f_2 y^2 + \ldots\ldots = \log\left(2e + e\mu y + \frac{e}{\mu y}\right) \\ &\quad + f_{-1}\frac{1}{y} + f_{-2}\frac{1}{y^2} + \ldots\ldots \\ &= 1 + \log(1 + \mu y) + \log\left(1 + \frac{1}{\mu y}\right) \\ &= 1 + \mu y - \tfrac{1}{2}\mu^2 y^2 + \tfrac{1}{3}\mu^3 y^3 - \ldots\ldots \\ &\qquad + \frac{1}{\mu y} - \tfrac{1}{2}\frac{1}{\mu^2 y^2} + \tfrac{1}{2}\frac{1}{\mu^3 y^3} - \ldots\ldots, \end{aligned}$$

and thus f_0, f_1, f_{-1}, f_2, f_{-2}, ... are determined, and we reach the relation

$$\begin{aligned} &\exp\begin{pmatrix} g_0 + g_1\mu - \tfrac{1}{2}g_2\mu^2 + \tfrac{1}{2}g_3\mu^3 - \ldots\ldots \\ + g_{-1}\frac{1}{\mu} - \tfrac{1}{2}g_{-2}\frac{1}{\mu^2} + \tfrac{1}{3}g_{-3}\frac{1}{\mu^3} - \ldots\ldots \end{pmatrix} \\ &\qquad\qquad = \overline{\exp}\left\{(2e-1)g_0 + eg_1\mu + eg_{-1}\frac{1}{\mu}\right\}. \end{aligned}$$

142. For present purposes it is useful to mention another theorem in regard to these operations.

Let P denote any linear function of the operators and consider the combination

$$\exp g_0 . \overline{\exp}\, P,$$

where $\overline{\exp}\, P$ and $\exp g_0$ denote two successive operations.

It is easy to prove that

$$\exp g_0 . \overline{\exp}\, P = \overline{\exp}\,\{eP + (e-1)g_0\},$$

and writing $P = g_0 + g_1\mu + g_{-1}\frac{1}{\mu}$, we have

$$\exp g_0 . \overline{\exp}\left(g_0 + g_1\mu + g_{-1}\frac{1}{\mu}\right) = \overline{\exp}\left\{(2e-1)g_0 + eg_1\mu + eg_{-1}\frac{1}{\mu}\right\},$$

and this leads to

$$\exp g_0 . \overline{\exp}\left(g_0 + g_1\mu + g_{-1}\frac{1}{\mu}\right) = \exp\begin{pmatrix} g_0 + g_1\mu - \tfrac{1}{2}g_2\mu^2 + \ldots\ldots \\ + g_{-1}\frac{1}{\mu} - \tfrac{1}{2}g_{-2}\frac{1}{\mu^2} + \ldots\ldots \end{pmatrix}$$

and thence to

$$\overline{\exp}\left(g_0 + g_1\mu + g_{-1}\frac{1}{\mu}\right) = \exp\begin{pmatrix} g_1\mu - \tfrac{1}{2}g_2\mu^2 + \tfrac{1}{3}g_3\mu^3 - \ldots\ldots \\ + g_{-1}\frac{1}{\mu} - \tfrac{1}{2}g_{-2}\frac{1}{\mu^2} + \tfrac{1}{3}g_{-3}\frac{1}{\mu^3} - \ldots\ldots \end{pmatrix}.$$

143. Recalling previous results, we now obtain

$$\begin{aligned}1+D_0+D_1\mu+D_2\mu^2+\dots\dots &= \exp\left\{d_0+d_1\mu-\tfrac{1}{2}d_2\mu^2+\dots\dots\right.\\ +D_{-1}\frac{1}{\mu}+D_{-2}\frac{1}{\mu^2}+\dots\dots &\qquad \left.+d_{-1}\frac{1}{\mu}-\tfrac{1}{2}d_{-2}\frac{1}{\mu^2}+\dots\dots\right\},\end{aligned}$$

$$\begin{aligned}1+G_0+G_1\mu+G_2\mu^2+\dots\dots &= \exp\left\{g_0+g_1\mu-\tfrac{1}{2}g_2\mu^2+\dots\dots\right.\\ +G_{-1}\frac{1}{\mu}+G_{-2}\frac{1}{\mu^2}+\dots\dots &\qquad \left.+g_{-1}\frac{1}{\mu}-\tfrac{1}{2}g_{-2}\frac{1}{\mu^2}+\dots\dots\right\},\end{aligned}$$

and from these

144.

$$\begin{aligned}(1+D_0) &= e^{d_0}\left\{1+d_1d_{-1}+\frac{d_1^2-d_2}{2!}\cdot\frac{d_{-1}^2-d_{-2}}{2!}\right.\\ &\qquad\left.+\frac{d_1^3-3d_2d_1+2d_3}{3!}\cdot\frac{d_{-1}^3-3d_{-2}d_{-1}+2d_{-3}}{3!}+\dots\dots\right\},\\ D_1 &= e^{d_0}\left\{d_1+\frac{d_1^2-d_2}{2!}\cdot d_{-1}+\frac{d_1^3-3d_2d_1+2d_3}{3!}\cdot\frac{d_{-1}^2-d_{-2}}{2!}+\dots\dots\right\},\\ D_{-1} &= e^{d_0}\left\{d_{-1}+d_1\frac{d_{-1}^2-d_{-2}}{2!}+\frac{d_1^2-d_2}{2!}\cdot\frac{d_{-1}^3-3d_{-2}d_{-1}+2d_{-3}}{3!}+\dots\dots\right\},\end{aligned}$$

etc. = etc.

$$\begin{aligned}1+G_0 &= e^{g_0}\left\{1+g_1g_{-1}+\frac{g_1^2-g_2}{2!}\cdot\frac{g_{-1}^2-g_{-2}}{2!}\right.\\ &\qquad\left.+\frac{g_1^3-3g_2g_1+2g_3}{3!}\cdot\frac{g_{-1}^3-3g_{-2}g_{-1}+2g_{-3}}{3!}+\dots\dots\right\},\\ G_1 &= e^{g_0}\left\{g_1+\frac{g_1^2-g_2}{2!}g_{-1}+\frac{g_1^3-3g_2g_1+2g_3}{3!}\cdot\frac{g_{-1}^2-g_{-2}}{2!}+\dots\dots\right\},\\ G_{-1} &= e^{g_0}\left\{g_{-1}+g_1\frac{g_{-1}^2-g_{-2}}{2!}+\frac{g_1^2-g_2}{2!}\cdot\frac{g_{-1}^3-3g_{-2}g_{-1}+2g_{-3}}{3!}+\dots\dots\right\},\end{aligned}$$

etc. = etc.

145. And also by taking logarithms

$$\begin{aligned}&d_0+d_1\mu-\tfrac{1}{2}d_2\mu^2+\tfrac{1}{3}d_3\mu^3-\dots\dots\\ &\quad+d_{-1}\frac{1}{\mu}-\tfrac{1}{2}d_{-2}\frac{1}{\mu^2}+\tfrac{1}{3}d_{-3}\frac{1}{\mu^3}-\dots\dots\\ &\qquad=\log\left\{1+D_0+D_1\mu+D_2\mu^2+\dots\dots\right.\\ &\qquad\qquad\left.+D_{-1}\frac{1}{\mu}+D_{-2}\frac{1}{\mu^2}+\dots\dots\right\},\end{aligned}$$

$$
\begin{aligned}
&g_0 + g_1\mu - \tfrac{1}{2} g_2\mu^2 + \tfrac{1}{3} g_3\mu^3 - \ldots\ldots \\
&\quad + g_{-1}\frac{1}{\mu} - \tfrac{1}{2} g_{-2}\frac{1}{\mu^2} + \tfrac{1}{3} g_{-3}\frac{1}{\mu^3} - \ldots\ldots \\
&\qquad = \log\Big\{1 + G_0 + G_1\mu + G_2\mu^2 + \ldots\ldots \\
&\qquad\qquad + G_{-1}\frac{1}{\mu} + G_{-2}\frac{1}{\mu^2} + \ldots\ldots\Big\},
\end{aligned}
$$

which lead to the relations

$$
\begin{aligned}
g_0 &= \sum \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\ldots\ldots\pi_2!\,\pi_1!\,\pi_0!\,\pi_{-1}!\,\pi_{-2}!\ldots\ldots}\ldots\ldots G_2^{\pi_2}G_1^{\pi_1}G_0^{\pi_0}G_{-1}^{\pi_{-1}}G_{-2}^{\pi_{-2}}\ldots, \\
\frac{(-)^{m-1}}{m}\,g_m &= \text{ditto}, \\
\frac{(-)^{m-1}}{m}\,g_{-m} &= \text{ditto},
\end{aligned}
$$

the summations having regard respectively to the solutions of the indeterminate equations

$$\Sigma t\pi_t = 0;\ \Sigma t\pi_t = m;\ \Sigma t\pi_t = -m.$$

We have also similar relations between the d and D operations, or if

$$\theta_1,\ \theta_2,\ \theta_3,\ \ldots\ldots;\ \phi_1,\ \phi_2,\ \phi_3,\ \ldots\ldots$$

be two sets of fictitious quantities such that

$$
\begin{aligned}
&\left(1 + d_1x + \frac{d_1^2 - d_2}{2} + \ldots\ldots\right)\left(1 + d_{-1}\frac{1}{x} + \frac{d_{-1}^2 - d_{-2}}{2}\frac{1}{x^2} + \ldots\ldots\right) \\
&\qquad = (1 + \theta_1x)(1 + \theta_2x)\ldots\ldots\left(1 + \frac{1}{\theta_1x}\right)\left(1 + \frac{1}{\theta_2x}\right)\ldots\ldots, \\
&\left(1 + g_1x + \frac{g_1^2 - g_2}{2} + \ldots\ldots\right)\left(1 + g_{-1}\frac{1}{x} + \frac{g_{-1}^2 - g_{-2}}{2}\frac{1}{x^2} + \ldots\ldots\right) \\
&\qquad = (1 + \phi_1x)(1 + \phi_2x)\ldots\ldots\left(1 + \frac{1}{\phi_1x}\right)\left(1 + \frac{1}{\phi_2x}\right)\ldots\ldots,
\end{aligned}
$$

and we represent symmetric functions of the two sets by partitions in brackets [ ] and [ ]′ respectively, we have the following correspondence between quantity and operations:

I.

Quantity.

146.

$$
\begin{aligned}
&e^n(1 + a_1x + a_2x^2 + \ldots\ldots)\left(1 + a_{-1}\frac{1}{x} + a_{-2}\frac{1}{x^2} + \ldots\ldots\right) \\
&\qquad = e^n(1 + \alpha_1x)(1 + \alpha_2x)\ldots\ldots\left(1 + \frac{1}{\alpha_1x}\right)\left(1 + \frac{1}{\alpha_2x}\right)\ldots\ldots \\
&\qquad = e^n\,\overline{\exp}\left\{(0) + (1)x + (\bar{1})\frac{1}{x}\right\},
\end{aligned}
$$

where $\frac{(1)^\lambda\,(0)^\mu\,(\bar{1})^\nu}{\lambda!\ \mu!\ \nu!}$ is symbolic expression for $(1^\lambda 0^\mu 1^\nu)$

$$\begin{aligned} &= 1 + A_0 + A_1 x + A_2 x^2 + \ldots . \\ &\qquad + A_{-1}\frac{1}{x} + A_{-2}\frac{1}{x^2} + \ldots . \\ &= \exp\left\{ (0) + (1)\,x - \tfrac{1}{2}(2)\,x^2 + \ldots . \right. \\ &\qquad \left. + (\bar{1})\frac{1}{x} - \tfrac{1}{2}(\bar{2})\frac{1}{x^2} + \ldots . \right\} \end{aligned}$$

II.

d-Operations.

147.

$$\begin{aligned} e^{d_0}\left(1 + d_1 x + \frac{d_1^2 - d_2}{2} + \ldots .\right)&\left(1 + d_{-1}\frac{1}{x} + \frac{d_{-1}^2 - d_{-2}}{2}\frac{1}{x^2} + \ldots .\right) \\ &= e^{d_0}(1 + \theta_1 x)(1 + \theta_2 x)\ldots .\left(1 + \frac{1}{\theta_1 x}\right)\left(1 + \frac{1}{\theta_2 x}\right)\ldots . \\ &= \overline{\exp}\left\{[0] + [1]\,x + [\bar{1}]\frac{1}{x}\right\}, \end{aligned}$$

where $\frac{[1]^\lambda\,[0]^\mu\,[\bar{1}]^\nu}{\lambda!\ \mu!\ \nu!}$ is symbolic expression for $[1^\lambda 0^\mu \bar{1}^\nu]$,

$$\begin{aligned} &= 1 + D_0 + D_1 x + D_2 x^2 + \ldots . \\ &\qquad + D_{-1}\frac{1}{x} + D_{-2}\frac{1}{x^2} + \ldots . \\ &= \exp\left\{ [0] + [1]\,x - \tfrac{1}{2}[2]\,x^2 + \ldots . \right. \\ &\qquad \left. + [\bar{1}]\frac{1}{x} - \tfrac{1}{2}[\bar{2}]\frac{1}{x^2} + \ldots . \right\} \end{aligned}$$

so that $$[m] = d_m,$$

and $$[p_1^{\pi_1} p_2^{\pi_2}\ldots .] = \frac{\overline{d_{p_1}^{\pi_1} d_{p_2}^{\pi_2}\ldots .}}{\pi_1!\ \pi_2!\ldots .}.$$

III.

g-Operations.

148.

$$\begin{aligned} e^{g_0}\left(1 + g_1 x + \frac{g_1^2 - g_2}{2} + \ldots .\right)&\left(1 + g_{-1}\frac{1}{x} + \frac{g_{-1}^2 - g_{-2}}{2}\frac{1}{x^2} + \ldots .\right) \\ &= e^{g_0}(1 + \phi_1 x)(1 + \phi_2 x)\ldots .\left(1 + \frac{1}{\phi_1 x}\right)\left(1 + \frac{1}{\phi_2 x}\right)\ldots . \\ &= \exp\,[0]'\ \overline{\exp}\left\{[0]' + [1]'\,x + [\bar{1}]'\frac{1}{x}\right\}, \end{aligned}$$

where $\frac{[1]'^{\lambda}[0]'^{\mu}[\bar{1}]'^{\nu}}{\lambda!\,\mu!\,\nu!}$ is a symbolic expression for $[1^{\lambda}0^{\mu}\bar{1}^{\nu}]'$,

$$\begin{aligned} &= 1 + G_0 + G_1x + G_2x^2 + \dots \\ &\qquad + G_{-1}\frac{1}{x} + G_{-2}\frac{1}{x^2} + \dots \\ &= \exp\left\{[0]' + [1]'x - \tfrac{1}{2}[2]'x^2 + \dots \right. \\ &\qquad \left. + [\bar{1}]'\frac{1}{x} - \tfrac{1}{2}[\bar{2}]'\frac{1}{x^2} + \dots\right\}, \end{aligned}$$

so that

$$[m]' = g_m,$$

$$[p_1^{\pi_1}p_2^{\pi_2}]' = \frac{\overline{g_{p_1}^{\pi_1}g_{p_2}^{\pi_2}\dots}}{\pi_1!\,\pi_2!\dots}.$$

§10.

149. I now apply the foregoing section to a new demonstration of the "Law of Reciprocity" in the theory of separations of which a purely arithmetical proof was given in the second memoir.

Consider three identities

$$\begin{aligned} &1 + A_0 + A_1x + A_2x^2 + \dots + A_{-1}\frac{1}{x} + A_{-2}\frac{1}{x^2} + \dots, \\ &\qquad = e^{n}(1+\alpha_1x)(1+\alpha_2x)\dots\left(1+\frac{1}{\alpha_1x}\right)\left(1+\frac{1}{\alpha_2x}\right)\dots \\ &1 + B_0 + B_1x + B_2x^2 + \dots + B_{-1}\frac{1}{x} + B_{-2}\frac{1}{x^2} + \dots, \\ &\qquad = e^{n'}(1+\beta_1x)(1+\beta_2x)\dots\left(1+\frac{1}{\beta_1x}\right)\left(1+\frac{1}{\beta_2x}\right)\dots, \\ &1 + C_0 + C_1x + C_2x^2 + \dots + C_{-1}\frac{1}{x} + C_{-2}\frac{1}{x^2} + \dots, \\ &\qquad = e^{n''}(1+\gamma_1x)(1+\gamma_2x)\dots\left(1+\frac{1}{\gamma_1x}\right)\left(1+\frac{1}{\gamma_2x}\right)^{*}\dots, \end{aligned}$$

and let symmetric functions of the quantities

$\alpha_1, \alpha_2, \alpha_3, \dots$ be denoted by partitions in $(\ )_\alpha$,
$\beta_1, \beta_2, \beta_3, \dots$ " " " " " $(\ )_\beta$,
$\gamma_1, \gamma_2, \gamma_3, \dots$ " " " " " $(\ )_\gamma$,

---

***n, n', n'' are each to be supposed indefinitely great, and further $n = n'$.**

then we have a triad of identities.

150.

$$\begin{aligned} 1 + A_0 + A_1 x + A_2 x^2 + \dots &= \exp\left\{ (0)_\alpha + (1)_\alpha x - \tfrac{1}{2}(2)_\alpha x^2 + \dots \right. \\ + A_{-1}\frac{1}{x} + A_{-2}\frac{1}{x^2} + \dots &\qquad \left. + (\bar{1})_\alpha \frac{1}{x} - \tfrac{1}{2}(\bar{2})_\alpha \frac{1}{x^2} + \dots \right\}, \end{aligned}$$

$$\begin{aligned} 1 + B_0 + B_1 x + B_2 x^2 + \dots &= \exp\left\{ (0)_\beta + (1)_\beta x - \tfrac{1}{2}(2)_\beta x^2 + \dots \right. \\ + B_{-1}\frac{1}{x} + B_{-2}\frac{1}{x^2} + \dots &\qquad \left. + (\bar{1})_\beta \frac{1}{x} - \tfrac{1}{2}(\bar{2})_\beta \frac{1}{x^2} + \dots \right\}, \end{aligned}$$

$$\begin{aligned} 1 + C_0 + C_1 x + C_2 x^2 + \dots &= \exp\left\{ (0)_\gamma + (1)_\gamma x - \tfrac{1}{2}(2)_\gamma x^2 + \dots \right. \\ + C_{-1}\frac{1}{x} + C_{-2}\frac{1}{x^2} + \dots &\qquad \left. + (\bar{1})_\gamma \frac{1}{x} - \tfrac{1}{2}(\bar{2})_\gamma \frac{1}{x^2} + \dots \right\}. \end{aligned}$$

151. Now assume that, between the quantities herein involved, there exists the relation

$$\begin{aligned} & 1 + C_0 + C_1 y + C_2 y^2 + \dots \\ & \quad + C_{-1}\frac{1}{y} + C_{-2}\frac{1}{y^2} + \dots \\ & \qquad = \prod_{s=1}^{s=n} \left(1 + B_0 + \alpha_s B_1 y + \alpha_s^2 B_2 y^2 + \dots \right. \\ & \qquad\qquad \left. + \frac{1}{\alpha_s} B_{-1} \frac{1}{y} + \frac{1}{\alpha_s^2} B_{-2} \frac{1}{y^2} + \dots \right), \end{aligned}$$

a relation which also implies the identity

$$1 + c_1 y + c_2 y^2 + c_3 y^3 + \dots = \prod_{s=1}^{s=n} (1 + \alpha_s b_1 y + \alpha_s^2 b_2 y^2 + \alpha_s^3 b_3 y^3 + \dots),$$

y being arbitrary, so that

$$\begin{aligned} & \log\left(1 + C_0 + C_1 y + C_2 y^2 + \dots \right. \\ & \qquad \left. + C_{-1}\frac{1}{y} + C_{-2}\frac{1}{y^2} + \dots \right) \\ & \qquad = \sum_{s=1}^{s=n} \log\left(1 + B_0 + \alpha_s B_1 y + \alpha_s^2 B_2 y^2 + \dots \right. \\ & \qquad\qquad \left. + \frac{1}{\alpha_s} B_{-1} \frac{1}{y} + \frac{1}{\alpha_s^2} B_{-2} \frac{1}{y^2} + \dots \right), \end{aligned}$$

leading to

$$\begin{aligned} (0)_\gamma + (1)_\gamma y - \tfrac{1}{2}(2)_\gamma y^2 + \dots &= (0)_\alpha (0)_\beta + (1)_\alpha (1)_\beta y - \tfrac{1}{2}(2)_\alpha (2)_\beta y^2 + \dots \\ + (\bar{1})_\gamma \frac{1}{y} - \tfrac{1}{2}(\bar{2})_\gamma \frac{1}{y^2} + \dots &\qquad + (\bar{1})_\alpha (\bar{1})_\beta \frac{1}{y} - \tfrac{1}{2}(\bar{2})_\alpha (\bar{2})_\beta \frac{1}{y^2} + \dots \end{aligned}$$

and thence to $$(m)_\gamma = (m)_\alpha (m)_\beta,$$

where m is any integer, positive, zero, or negative.

152. This result, which is of great importance, shows that the function of C_0, C_1, C_{-1}, C_2, C_{-2}, ..., denoted by $(m)_\gamma$, is unaltered when the n quantities $\alpha_1, \alpha_2, \alpha_3, \dots$ and the several n' quantities $\beta_1, \beta_2, \beta_3, \dots$ are interchanged; but every symmetric function is expressible in terms of sums of powers of the quantities, and it hence follows that every symmetric function of the n quantities

$$\gamma_1, \gamma_2, \gamma_3, \dots$$

remains unaltered by the interchange of the n quantities

$$\alpha_1, \alpha_2, \alpha_3, \dots$$

with the several n' quantities

$$\beta_1, \beta_2, \beta_3, \dots;$$

we may say, in fact, that if any assemblage of partitions in brackets

$$(\quad)_\gamma$$

be expressed in terms of partitions in brackets

$$(\quad)_\alpha \text{ and } (\quad)_\beta,$$

it remains unaltered by the interchange of the brackets $(\quad)_\alpha$ and $(\quad)_\beta$.

153. For example, it is shown in this way that if we have a result

$$(s_1^{\sigma_1}s_2^{\sigma_2}\dots)_\gamma = \dots + J(p_1^{\pi_1}p_2^{\pi_2}\dots)_\alpha(\lambda_1^{l_1}\lambda_2^{l_2}\dots)_\beta + \dots$$

we must also have

$$(s_1^{\sigma_1}s_2^{\sigma_2}\dots)_\gamma = \dots + J\{(p_1^{\pi_1}p_2^{\pi_2}\dots)_\alpha(\lambda_1^{l_1}\lambda_2^{l_2}\dots)_\beta + (p_1^{\pi_1}p_2^{\pi_2}\dots)_\beta(\lambda_1^{l_1}\lambda_2^{l_2}\dots)_\alpha\} + \dots,$$

and this fact will be shown to involve the "Law of Reciprocity" brought forward in the first and second memoirs.

154. It should be remarked that, as a consequence, the assumed relation may be also written in the form

$$\begin{aligned} 1 + C_0 + C_1 y + C_2 y^2 + \dots& \\ + C_{-1}\frac{1}{y} + C_{-2}\frac{1}{y^2} + \dots& \\ = \prod_{s=1}^{s=n}\Bigg(1 + A_0 + \beta_s A_1 y + \beta_s^2 A_2 y^2 &+ \dots \\ + \frac{1}{\beta_s}A_{-1}\frac{1}{y} + \frac{1}{\beta_s^2}A_{-2}\frac{1}{y^2} &+ \dots\Bigg). \end{aligned}$$

155. Associated with the triad of identities above set forth, we have a triad of operator relations, which I will write as follows:

$$\begin{aligned}1 &+ {}_\alpha G_0 + {}_\alpha G_1 y + {}_\alpha G_2 y^2 + \dots\\ &+ {}_\alpha G_{-1}\frac{1}{y} + {}_\alpha G_{-2}\frac{1}{y^2} + \dots\\ &= \exp\left\{\begin{aligned}&[0]'_\alpha + [1]'_\alpha y - \tfrac{1}{2}[2]'_\alpha y^2 + \dots\\ &+ [\bar{1}]'_\alpha\frac{1}{y} - \tfrac{1}{2}[\bar{2}]'_\alpha\frac{1}{y^2} + \dots\end{aligned}\right\},\end{aligned}$$

where
$$[m]'_\alpha = {}_\alpha g_m,$$
$$[p_1^{\pi_1}p_2^{\pi_2}\dots]'_\alpha = \frac{\overline{{}_\alpha g_{p_1}^{\pi_1}\,{}_\alpha g_{p_2}^{\pi_2}}\dots}{\pi_1!\,\pi_2!\dots},$$

$$\begin{aligned}1 &+ {}_\beta G_0 + {}_\beta G_1 y + {}_\beta G_2 y^2 + \dots\\ &+ {}_\beta G_{-1}\frac{1}{y} + {}_\beta G_{-2}\frac{1}{y^2} + \dots\\ &= \exp\left\{\begin{aligned}&[0]'_\beta + [1]'_\beta y - \tfrac{1}{2}[2]'_\beta y^2 + \dots\\ &+ [\bar{1}]_\beta\frac{1}{y} - \tfrac{1}{2}[\bar{2}]'_\beta\frac{1}{y^2} + \dots\end{aligned}\right\},\end{aligned}$$

where
$$[m]'_\beta = {}_\beta g_m,$$
$$[p_1^{\pi_1}p_2^{\pi_2}\dots]'_\beta = \frac{\overline{{}_\beta g_{p_1}^{\pi_1}\,{}_\beta g_{p_2}^{\pi_2}}\dots}{\pi_1!\,\pi_2!\dots},$$

$$\begin{aligned}1 &+ {}_\gamma G_0 + {}_\gamma G_1 y + {}_\gamma G_2 y^2 + \dots\\ &+ {}_\gamma G_{-1}\frac{1}{y} + {}_\gamma G_{-2}\frac{1}{y^2} + \dots\\ &= \exp\left\{\begin{aligned}&[0]'_\gamma + [1]'_\gamma y - \tfrac{1}{2}[2]'_\gamma y^2 + \dots\\ &+ [\bar{1}]'_\gamma\frac{1}{y} - \tfrac{1}{2}[\bar{2}]'_\gamma\frac{1}{y^2} + \dots\end{aligned}\right\},\end{aligned}$$

where
$$[m]'_\gamma = {}_\gamma g_m,$$
$$[p_1^{\pi_1}p_2^{\pi_2}\dots]'_\gamma = \frac{\overline{{}_\gamma g_{p_1}^{\pi_1}\,{}_\gamma g_{p_2}^{\pi_2}}\dots}{\pi_1!\,\pi_2!\dots}.$$

156. Now writing the assumed relation, viz.

$$\begin{aligned}1 &+ C_0 + C_1 y + C_2 y^2 + \dots\\ &+ C_{-1}\frac{1}{y} + C_{-2}\frac{1}{y^2} + \dots\\ &= \prod_{s=1}^{s=n}\left(\begin{aligned}&1 + B_0 + \alpha_s B_1 y + \alpha_s^2 B_2 y^2 + \dots\\ &+ \frac{1}{\alpha_s}B_{-1}\frac{1}{y} + \frac{1}{\alpha_s^2}B_{-2}\frac{1}{y^2} + \dots\end{aligned}\right)\end{aligned}$$

in the abbreviated form

$$U = u_{\alpha_1} u_{\alpha_2} u_{\alpha_3} \ldots ,$$

we have

$$ {}_\beta g_m U = ({}_\beta g_m u_{\alpha_1}) u_{\alpha_2} u_{\alpha_3} \ldots + u_{\alpha_1} ({}_\beta g_m u_{\alpha_2}) u_{\alpha_3} \ldots + u_{\alpha_1} u_{\alpha_2} ({}_\beta g_m u_{\alpha_3}) \ldots + \ldots $$

and

$$ \begin{aligned} {}_\beta g_m u_{\alpha_1} &= \left\{ \begin{array}{l} (1+B_0)\partial_{B_m} + B_1\partial_{B_{m+1}} + \ldots \\ \qquad\qquad + B_{-1}\partial_{B_{m-1}} + \ldots \end{array} \right\} \left(\begin{array}{l} 1 + B_0 + \alpha_1 B_1 y + \alpha_1^2 B_2 y^2 + \ldots \\ \qquad + \dfrac{1}{\alpha_1} B_{-1} \dfrac{1}{y} + \dfrac{1}{\alpha_1^2} B_{-2} \dfrac{1}{y^2} + \ldots \end{array} \right) \\ &= \alpha_1^m y^m u_{\alpha_1} ; \end{aligned} $$

hence $\qquad {}_\beta g_m U = (\alpha_1^m + \alpha_2^m + \alpha_3^m + \ldots) y^m U = (m)_\alpha y^m U,$

leading to

$$ \begin{aligned} &{}_\beta g_m (1 + C_0) + y\, {}_\beta g_m C_1 + y^2\, {}_\beta g_m C_2 + \ldots \\ &\qquad + \frac{1}{y}\, {}_\beta g_m C_{-1} + \frac{1}{y^2}\, {}_\beta g_m C_{-2} + \ldots \\ &\qquad\qquad = (m)_\alpha y^m \left\{ \begin{array}{l} 1 + C_0 + C_1 y + C_2 y^2 + \ldots \\ \qquad + C_{-1}\dfrac{1}{y} + C_{-2}\dfrac{1}{y^2} + \ldots \end{array} \right\} , \end{aligned} $$

which gives, on equating coefficients of like powers of y,

$$ \begin{aligned} {}_\beta g_m (1 + C_0) &= (m)_\alpha C_{-m} , \\ {}_\beta g_m C_1 &= (m)_\alpha C_{-m+1} , \\ {}_\beta g_m C_{-1} &= (m)_\alpha C_{-m-1} , \\ &\ldots\ldots\ldots\ldots \\ {}_\beta g_m C_m &= (m)_\alpha (1 + C_0) ; \end{aligned} $$

but regarding $B_0, B_1, B_{-1}, B_2, B_{-2}, \ldots$ as functions of $C_0, C_1, C_{-1}, C_2, C_{-2}, \ldots$ only, we have

$$ \begin{aligned} {}_\beta g_m &\equiv {}_\beta g_m (1 + C_0)\partial_{C_0} + {}_\beta g_m C_1 \partial_{C_1} + {}_\beta g_m C_2 \partial_{C_2} + \ldots \\ &\qquad\qquad + {}_\beta g_m C_{-1} \partial_{C_{-1}} + {}_\beta g_m C_{-2} \partial_{C_{-2}} + \ldots \\ &= (m)_\alpha \left\{ \begin{array}{l} C_{-m}\partial_{C_0} + C_{-m+1}\partial_{C_1} + C_{-m+2}\partial_{C_2} + \ldots \\ \qquad\qquad + C_{-m-1}\partial_{C_{-1}} + C_{-m-2}\partial_{C_{-2}} + \ldots \end{array} \right\} , \end{aligned} $$

or $\qquad {}_\beta g_m = (m)_\alpha\, {}_\gamma g_m ;$

we thus arrive at the conclusion that, regarding the assumed relation as defining a transformation of any function of $C_0, C_1, C_{-1}, C_2, C_{-2}, \ldots$ into a function of $B_0, B_1, B_{-1}, B_2, B_{-2}, \ldots\ldots, \alpha_1, \alpha_2, \alpha_3, \ldots\ldots$ being the constants of the transformation, the operation

$$_\gamma g_m$$

is an invariant.

157. It may be remarked by the way that

$$_\gamma g_m \log \left\{ \begin{array}{l} 1 + C_0 + C_1 x \quad + C_2 x^2 \quad + \ldots\ldots \\ \qquad + C_{-1}\dfrac{1}{x} + C_{-2}\dfrac{1}{x^2} + \ldots\ldots \end{array} \right\} = x^m,$$

whence

$$_\gamma g_m \left\{ \begin{array}{l} (0)_\gamma + (1)_\gamma x \quad - \tfrac{1}{2}(2)_\gamma x^2 \; + \ldots\ldots \\ \qquad + (\overline{1})_\gamma \dfrac{1}{x} - \tfrac{1}{2}(\overline{2})\dfrac{1}{x^2} + \ldots\ldots \end{array} \right\} = x^m,$$

that is,

$$_\gamma g_m \frac{(-)^{m+1}}{m}(m)_\gamma = 1,$$

$$_\gamma g_{-m} \frac{(-)^{m+1}}{m}(\overline{m})_\gamma = 1,$$

where m differs from zero, and

$$_\gamma g_0 (0)_\gamma = 1,$$

whilst

$$_\gamma g_m (s)_\gamma = 0 \text{ if } s \neq m;$$

we have thus a set of transcendental solutions of the partial differential equation

$$_\gamma g_m = 0,$$

and all of these solutions have otherwise been proved to be invariants of the supposed transformation (vide second memoir).

158. Returning to the relation

$$_\beta g_m = (m)_\alpha\, {}_\gamma g_m,$$

we may write

$$\begin{array}{l} _\beta g_0 + {}_\beta g_1 y \quad - \tfrac{1}{2}\,{}_\beta g_2 y^2 \quad + \ldots\ldots = (0)_\alpha + (1)_\alpha\, {}_\gamma g_1 y \quad - \tfrac{1}{2}(2)_\alpha\, {}_\gamma g_2 y^2 \quad + \ldots\ldots \\ \quad + {}_\beta g_{-1}\dfrac{1}{y} - \tfrac{1}{2}\,{}_\beta g_{-2}\dfrac{1}{y^2} + \ldots\ldots \qquad\quad + (\overline{1})_\alpha\, {}_\gamma g_{-1}\dfrac{1}{y} - \tfrac{1}{2}(\overline{2})_\alpha\, {}_\gamma g_{-2}\dfrac{1}{y^2} + \ldots\ldots, \end{array}$$

y being arbitrary, and also in the form

$$\exp \left\{ \begin{array}{l} [0]'_\beta + [1]'_\beta y \quad - \tfrac{1}{2}[2]'_\beta y^2 \quad + \ldots\ldots \\ \qquad + [\overline{1}]'_\beta \dfrac{1}{y} - \tfrac{1}{2}[\overline{2}]'_\beta \dfrac{1}{y^2} + \ldots\ldots \end{array} \right\},$$

$$= \exp \left\{ \begin{array}{l} (0)_\alpha [0]'_\gamma + (1)_\alpha [1]'_\gamma y \quad - \tfrac{1}{2}(2)_\alpha [2]'_\gamma y^2 \quad + \ldots\ldots \\ \qquad + (\overline{1})_\alpha [\overline{1}]'_\gamma \dfrac{1}{y} - \tfrac{1}{2}(\overline{2})_\alpha [\overline{2}]'_\gamma \dfrac{1}{y^2} + \ldots\ldots \end{array} \right\},$$

and then from previous work we are led to the result

159.

$$\begin{aligned}1 + {}_{\beta}G_0 + {}_{\beta}G_1 y + {}_{\beta}G_2 y^2 + \dots\\ + {}_{\beta}G_{-1}\frac{1}{y} + {}_{\beta}G_{-2}\frac{1}{y^2} + \dots\\ = \prod_{s=1}^{s=n}\left(1 + {}_{\gamma}G_0 + \alpha_s\,{}_{\gamma}G_1 y + \alpha_s^2\,{}_{\gamma}G_2 y^2 + \dots\right.\\ \left. + \frac{1}{\alpha_s}\,{}_{\gamma}G_{-1}\frac{1}{y} + \frac{1}{\alpha_s^2}\,{}_{\gamma}G_{-2}\frac{1}{y^2} + \dots\right),\end{aligned}$$

and a comparison with the assumed relation

$$\begin{aligned}1 + C_0 + C_1 y + C_2 y^2 + \dots\\ + C_{-1}\frac{1}{y} + C_{-2}\frac{1}{y^2} + \dots\\ = \prod_{s=1}^{s=n}\left(1 + B_0 + \alpha_s B_1 y + \alpha_s^2 B_2 y^2 + \dots\right.\\ \left. + \frac{1}{\alpha_s} B_{-1}\frac{1}{y} + \frac{1}{\alpha_s^2} B_{-2}\frac{1}{y^2} + \dots\right)\end{aligned}$$

leads to the following theorem:

160. "In any relation connecting the quantities

$$C_0,\ C_1,\ C_{-1},\ C_2,\ C_{-2},\ \dots$$

with the quantities

$$B_0,\ B_1,\ B_{-1},\ B_2,\ B_{-2},\ \dots$$

we are at liberty to substitute

$${}_{\beta}G_{\kappa} \text{ for } C_{\kappa}$$

and

$${}_{\gamma}G_{\kappa} \text{ for } B_{\kappa},$$

and we so obtain a relation between operators."

This very important theorem can be applied forthwith.

161. By means of the initial relation

$$\begin{aligned}1 + C_0 + C_1 x + C_2 x^2 + \dots\\ + C_{-1}\frac{1}{x} + C_{-2}\frac{1}{x^2} + \dots\\ = \prod_{s=1}^{s=n}\left(1 + B_0 + \alpha_s B_1 x + \alpha_s^2 B_2 x^2 + \dots\right.\\ \left. + \frac{1}{\alpha_s} B_{-1}\frac{1}{x} + \frac{1}{\alpha_s^2} B_{-2}\frac{1}{x^2} + \dots\right).\end{aligned}$$

Where x is arbitrary,* we can express C_0, C_1, C_{-1}, C_2, C_{-2}, ... in terms of the quantities B_0, B_1, B_{-1}, B_2, B_{-2}, ... and monomial symmetric functions of the n quantities

$$\alpha_1,\ \alpha_2,\ \ldots.\ \alpha_n;$$

and, multiplying out, we obtain a result such as

$$C_{p_1}^{\pi_1} C_{p_2}^{\pi_2} \ldots. = \ldots. + LB_{s_1}^{\sigma_1} B_{s_2}^{\sigma_2} \ldots. + \ldots.\,, \qquad \text{(I)}$$

and also a result such as

$$C_{\lambda_1}^{l_1} C_{\lambda_2}^{l_2} \ldots. = \ldots. + MB_{s_1}^{\sigma_1} B_{s_2}^{\sigma_2} \ldots. + \ldots. \qquad \text{(II)}$$

Join to these two, a third, viz.

$$(s_1^{\sigma_1} s_2^{\sigma_2} \ldots.)_\gamma = \ldots. + A\,(p_1^{\pi_1} p_2^{\pi_2} \ldots.)_\beta + B\,(\lambda_1^{l_1} \lambda_2^{l_2} \ldots.)_\beta + \ldots. \qquad \text{(III)}$$

Now the equation (I) yields the equation of operators†

$${}_\beta G_{p_1}^{\pi_1}\, {}_\beta G_{p_2}^{\pi_2} \ldots. = \ldots. + L\, {}_\gamma G_{s_1}^{\sigma_1}\, {}_\gamma G_{s_2}^{\sigma_2} \ldots. + \ldots.\,,$$

and performing each side of this upon the opposite side of the relation (III), we obtain

$$L\, {}_\gamma G_{s_1}^{\sigma_1}\, {}_\gamma G_{s_2}^{\sigma_2} \ldots. (s_1^{\sigma_1} s_2^{\sigma_2} \ldots.)_\gamma = A\, {}_\beta G_{p_1}^{\pi_1}\, {}_\beta G_{p_2}^{\pi_2} \ldots. (p_1^{\pi_1} p_2^{\pi_2} \ldots.)_\beta,$$

no other terms surviving the operation; but

$${}_\gamma G_{s_1}^{\sigma_1}\, {}_\gamma G_{s_2}^{\sigma_2} \ldots. (s_1^{\sigma_1} s_2^{\sigma_2} \ldots.)_\gamma = {}_\beta G_{p_1}^{\pi_1}\, {}_\beta G_{p_2}^{\pi_2} \ldots. (p_1^{\pi_1} p_2^{\pi_2} \ldots.)_\beta = 1;$$

hence $$L = A,$$

and also from the identities (II) and (III) we obtain

$$M = B,$$

and this leads to

$$(s_1^{\sigma_1} s_2^{\sigma_2} \ldots.)_\gamma = \ldots. + L\,(p_1^{\pi_1} p_2^{\pi_2} \ldots.)_\beta + M\,(\lambda_1^{l_1} \lambda_2^{l_2} \ldots.)_\beta + \ldots.$$

Now, it has been shown previously that

$$(s_1^{\sigma_1} s_2^{\sigma_2} \ldots)_\gamma = \ldots + J\left\{(\lambda_1^{l_1} \lambda_2^{l_2} \ldots)_\alpha (p_1^{\pi_1} p_2^{\pi_2} \ldots)_\beta + (\lambda_1^{l_1} \lambda_2^{l_2} \ldots)_\beta (p_1^{\pi_1} p_2^{\pi_2} \ldots)_\alpha\right\} + \ldots;$$

---

[*] This relation takes the place of the relation in the second memoir, viz.

$$\begin{aligned} 1 + X_0 + X_1 x + X_2 x^2 + \ldots. &= \prod_{s=1}^{s=n} \Big(1 + x_0 + a_s x_1 x + a_s^2 x_2 x^2 + \ldots. \\ + X_{-1}\frac{1}{x} + X_{-2}\frac{1}{x^2} + \ldots. &\qquad + \frac{1}{a_s} x_{-1} \frac{1}{x} + \frac{1}{a_s^2} x_{-2} \frac{1}{x^2} + \ldots.\Big), \end{aligned}$$

the notation alone being changed.

[†] Compare Hammond, loc. cit.

hence we must have

$$\begin{aligned} L &= \ldots . + J(\lambda_1^{l_1}\lambda_2^{l_2}\ \ldots .)_\alpha + \ldots . , \\ M &= \ldots . + J(p_1^{\pi_1}p_2^{\pi_2}\ldots .)_\alpha + \ldots . , \end{aligned}$$

162. And then the relations (I) and (II) become

$$\begin{aligned} C_{p_1}^{\pi_1}C_{p_2}^{\pi_2}\ldots . &= \ldots . + J(\lambda_1^{l_1}\lambda_2^{l_2}\ \ldots .)_\alpha B_{s_1}^{\sigma_1}B_{s_2}^{\sigma_2}\ldots . + \ldots . , \\ C_{\lambda_1}^{l_1}C_{\lambda_2}^{l_2}\ldots . &= \ldots . + J(p_1^{\pi_1}p_2^{\pi_2}\ldots .)_\alpha B_{s_1}^{\sigma_1}B_{s_2}^{\sigma_2}\ldots . + \ldots . , \end{aligned}$$

which is the law of reciprocity it was required to establish.

§11.

New Law of Symmetry.

163. Suppose that we find the relation

$$(p_1^{\pi_1}p_2^{\pi_2}\ldots .)_\gamma = \ldots . + PB_{s_1}^{\sigma_1}B_{s_2}^{\sigma_2}\ldots . + \ldots . ,$$

leading to the operator relation

$$\frac{\overline{{}_\beta g_{p_1}^{\pi_1}\,{}_\beta g_{p_2}^{\pi_2}}\ldots .}{\pi_1!\ \pi_2!\ldots .} = \ldots . + P\,{}_\gamma G_{s_1}^{\sigma_1}\,{}_\gamma G_{s_2}^{\sigma_2}\ldots . + \ldots . ,$$

where P is an aggregate of symmetric functions of the quantities $\alpha_1,\ \alpha_2, \ldots \alpha_n$.

Further, suppose that

$$(s_1^{\sigma_1}s_2^{\sigma_2}\ldots .)_\gamma = \ldots . +\ QB_{p_1}^{\pi_1}B_{p_2}^{\pi_2}\ldots . + \ldots . ,$$

since

$$\begin{aligned} {}_\beta g_m = \partial_{B_m} + B_0\partial_{B_m} &+ B_1\partial_{B_{m+1}} + \ldots . \\ &+ B_{-1}\partial_{B_{m-1}} + \ldots . , \end{aligned}$$

we have $\dfrac{\overline{{}_\beta g_{p_1}^{\pi_1}\,{}_\beta g_{p_2}^{\pi_2}}\ldots .}{\pi_1!\ \pi_2!\ldots .} = \dfrac{\partial_{B_{p_1}}^{\pi_1}\ \partial_{B_{p_2}}^{\pi_2}\ldots .}{\pi_1!\ \pi_2!\ldots .} +$ terms which, operating upon a function of B_0, B_1, B_{-1}, …., do not diminish its degree; hence, attending only to terms of like weight and degree,

$$P\,{}_\gamma G_{s_1}^{\sigma_1}\,{}_\gamma G_{s_2}^{\sigma_2}\ldots . (s_1^{\sigma_1}s_2^{\sigma_2}\ldots .)_\gamma = Q\,\frac{\partial_{B_{p_1}}^{\pi_1}\ \partial_{B_{p_2}}^{\pi_2}\ldots .}{\pi_1!\ \pi_2!\ldots .}\,B_{p_1}^{\pi_1}B_{p_2}^{\pi_2}\ldots . ,$$

that is, $P = Q$; therefore we have the theorem

164. "If

$$(p_1^{\pi_1}p_2^{\pi_2}\ \ldots .)_\gamma = \ldots . + A\,(\lambda_1^{l_1}\lambda_2^{l_2}\ldots .)_\alpha B_{s_1}^{\sigma_1}B_{s_2}^{\sigma_2}\ldots . + \ldots . ,$$

then

$$(s_1^{\sigma_1}s_2^{\sigma_2}\ldots .)_\gamma\ = \ldots . + A\,(\lambda_1^{l_1}\lambda_2^{l_2}\ldots .)_\alpha B_{p_1}^{\pi_1}B_{p_2}^{\pi_2}\ldots . + \ldots . \text{"}$$

165. To make manifest the importance of this theorem I take a concrete case and, for simplicity, restrict myself to symmetric functions which are expressible by means of partitions composed merely of positive integers. We have now, as arguments, the quantities

$$b_1,\ b_2,\ b_3,\ \ldots\ldots;\ c_1,\ c_2,\ c_3,\ \ldots\ldots$$

derived from the relations

$$1 + b_1x + b_2x^2 + \ldots\ldots = (1 + \beta_1 x)(1 + \beta_2 x)(1 + \beta_3 x)\ldots\ldots,$$
$$1 + c_1x + c_2x^2 + \ldots\ldots = (1 + \gamma_1 x)(1 + \gamma_2 x)(1 + \gamma_3 x)\ldots\ldots,$$

and then the fundamental relation leads to the set of identities

$$c_1 = (1)_\alpha b_1,$$
$$c_2 = (2)_\alpha b_2 + (1^2)_\alpha b_1^2,$$
$$c_3 = (3)_\alpha b_3 + (21)_\alpha b_2 b_1 + (1^3)_\alpha b_1^3,$$
$$c_4 = (4)_\alpha b_4 + (31)_\alpha b_3 b_1 + (2^2)_\alpha b_2^2 + (21^2)_\alpha b_2 b_1^2 + (1^4)_\alpha b_1^4,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

and we can, for example, form a table of the fourth order by expressing the symmetric functions of $\gamma_1, \gamma_2, \gamma_3, \ldots\ldots$ of weight four in terms of the quantities $b_1, b_2, b_3, b_4, \ldots\ldots$

166. Such a table is now given:

	b_4	b_3b_1	b_2^2	$b_2b_1^2$	b_1^4
$(4)_\gamma$	$-4(4)$	$4(3)(1)-4(31)$	$2(2)^2-4(2^2)$	$-4(2)(1)^2+4(2)(1^2)$ $+4(21)(1)\quad-4(21^2)$	$(1)^4-4(1^2)(1)^2+2(1^2)^2$ $+4(1^3)(1)\quad-4(1^4)$
$(31)_\gamma$	$4(4)$	$-\quad(3)(1)+4(31)$	$-2(2)^2+4(2^2)$	$(2)(1)^2-\quad(21)(1)$ $-4(2)(1^2)+4(21^2)$	$(1)^2(1^2)-\quad(1^3)(1)$ $-2(1^2)^2\quad+4(1^4)$
$(2^2)_\gamma$	$2(4)$	$-2(3)(1)+2(31)$	$(2)^2+2(2^2)$	$2(2)(1^2)-2(21)(1)$ $+2(21^2)$	$(1^2)^2-2(1^3)(1)$ $+2(1^4)$
$(21^2)_\gamma$	$-4(4)$	$(3)(1)-4(31)$	$-4(2^2)$	$(21)(1)-4(21^2)$	$(1^3)(1)-4(1^4)$
$(1^4)_\gamma$	(4)	(31)	(2^2)	(21^2)	(1^4)

This is to be read from left to right; for instance, the last line is read

$$(1^4)_\gamma = (4)\,b_4 + (31)\,b_3b_1 + (2^2)\,b_2^2 + (21^2)\,b_2b_1^2 + (1^4)\,b_1^4.$$

167. In this table the suffix α has been for convenience omitted.

The theorem shows that the s^{th} row and the s^{th} column are identical; the table in fact possesses, what is termed, row and column symmetry.

In any column the terms are all separations of the partition of the *B*-product at the head of the column.

In any row the assemblages of separations are formed according to a law defined by the γ partition at the left of the row.

168. To explain this I form the assemblage of separations of (21^2) according to the law defined by the partition (31); the process consists in first writing down the expression of (31) in terms of separations of (1^4), thus:

$$(31) = (1^2)(1)^2 - 2(1^2)^2 - (1^3)(1) + 4(1^4),$$

the specification of each term is then written down and, beneath, the corresponding coefficient, thus:

$$\left.\begin{array}{lcccc} \text{Specifications,} \;.\;.\;.\;. & (21^2) & (2^2) & (31) & (4) \\ \text{Coefficients,} \;.\;.\;.\;.\;. & +1 & -2 & -1 & +4 \end{array}\right\}.$$

The two lines, last written, define the law which is applied to the case in hand as follows: The line of specifications is again written down, and underneath each specification those separations of (21^2) which are of that specification, care being taken to write down a separation in correspondence with each permutation amongst separates of the same weight.

We thus obtain two lines, viz.

Specifications.	(21^2)	(2^2)	(31)	(4)
Separations.	$(2)(1)^2$	$(2)(1^2)$ $(1^2)(2)$	$(21)(1)$	(21^2)

We finally attach the coefficients, which appear in the definition of the law, to the separations of corresponding specification; the result is the assemblage

$$\begin{aligned} &(2)(1)^2 - 2(2)(1^2) - (21)(1) + 4(21^2) \\ &\qquad - 2(1^2)(2), \end{aligned}$$

or

$$(2)(1)^2 - 4(2)(1^2) - (21)(1) + 4(21^2),$$

which will be found in the second row and fourth column of the above table.

169. The process is conveniently placed in four rows as follows:

(31)	Specifications,	(21^2)	(2^2)	(31)	(4)
	Coefficients,	$+1$	-2	-1	$+4$
(21^2)	Separations,	$(2)(1)^2$	$(2)(1^2)$ $(1^2)(2)$	$(21)(1)$	(21^2)
	Assemblage,	$(2)(1)^2$	$-4(2)(1^2)$	$-(21)(1)$	$+4(21^2)$

Now, observe that the assemblage which occurs in the fourth row and second column is identical with that just found, and moreover is formed from separations of (31) according to the law defined by the partition (21^2), since

$$(21^2)=(1^3)(1)-4(1^4).$$

The process is as under:

(21^2)	Specifications,	(31)	(4)
	Coefficients,	$+1$	-4
(31)	Separations,	$(3)(1)$	(31)
	Assemblage,	$(3)(1)$	$-4(31)$

and the assemblage thus found is necessarily in the fourth row and second column of the table.

170. I now enunciate a law of symmetry.

Theorem:

"If any symmetric function $(p_1^{\pi_1}p_2^{\pi_2}\ldots)$ of weight n be expressed in terms of separations of the symmetric function (1^n), a term of specification $(\lambda_1^{l_1}\lambda_2^{l_2}\ldots)$ is found attached to a certain numerical coefficient x; two rows of specifications and corresponding numerical coefficients respectively are said to define the law of the symmetric function $(p_1^{\pi_1} p_2^{\pi_2} \dots .)$.' The assemblage of separations of a symmetric function $(s_1^{\sigma_1} s_2^{\sigma_2} \dots .)$, formed according to the law of the symmetric function $(p_1^{\pi_1} p_2^{\pi_2} \dots .)$ is identical with the assemblage, of separations of the symmetric function $(p_1^{\pi_1} p_2^{\pi_2} \dots .)$, formed according to the law of the symmetric function $(s_1^{\sigma_1} s_2^{\sigma_2} \dots .)$."

171. I subjoin an additional example of the theorem, indicating the reciprocity between the functions

(31^3) and (2^21^2) of weight 6.

$$(2^21^2) = (1^4)(1^2) \quad - 4(1^5)(1) \quad + 9(1^6),$$

(2^21^2)	Specifications,	(42)	(51)	(6)
	Coefficients,	$+1$	-4	$+9$
(31^3)	Separations,	$(31)(1^2)$	$(31^2)(1)$	(31^3)
	Assemblage,	$(31)(1^2)$	$-4(31^2)(1)$	$+9(31^3)$

$$(31^3) = (1^4)(1)^2 \quad - 2(1^4)(1^2) \quad - (1^5)(1) \quad + 6(1^6)$$

(31^3)	Specifications,	(41^2)	(42)	(51)	(6)
	Coefficients,	$+1$	-2	-1	$+6$
(2^21^2)	Separations,	$(2^2)(1)^2$	$(2^2)(1^2)$ $(21^2)(2)$	$(2^21)(1)$	(2^21^2)
	Assemblage,	$(2^2)(1)^2$	$-2(2^2)(1^2)$ $-2(21^2)(2)$	$-(2^21)(1)$	$+6(2^21^2)$

and the identity of the two assemblages, viz.

$$(31)(1^2) - 4(31^2)(1) + 9(31^3) = (2^2)(1)^2 - 2(2^2)(1^2) - 2(21^2)(2) - (2^21)(1) + 6(2^21^2)$$

is easily established; each is, in fact, equal to

$$(42) - 2(41^2) - 3(321).$$

172. The theorem is not restricted to positive integers, but there seems to be no advantage to be gained practically in extending it to the most general case.

§12.

173. In the foregoing section another law of symmetry was incidentally met with. As it is of importance, I propose to further examine it.

From the relation

$$C^{\pi_1}_{p_1} C^{\pi_2}_{p_2} \ldots\ldots = \ldots\ldots + LB^{\sigma_1}_{s_1} B^{\sigma_2}_{s_2} \ldots\ldots + \ldots\ldots, \qquad \text{(I)}$$

is derived the operator relation

$${}_\beta G^{\pi_1}_{p_1}\, {}_\beta G^{\pi_2}_{p_2} \ldots\ldots = \ldots\ldots + L\, {}_\gamma G^{\sigma_1}_{s_1}\, {}_\gamma G^{\sigma_2}_{s_2} \ldots\ldots + \ldots\ldots,$$

and operating with these two sides upon opposite sides of an *assumed* relation

$$(s_1^{\sigma_1} s_2^{\sigma_2} \ldots\ldots)_\gamma = \ldots\ldots + A\,(p_1^{\pi_1} p_2^{\pi_2} \ldots\ldots)_\beta + \ldots\ldots \qquad \text{(II)}$$

we obtain at once

$$L = A,$$

so that a law of symmetry is involved in the two results

$$\begin{cases} C^{\pi_1}_{p_1} C^{\pi_2}_{p_2} \ldots\ldots = \ldots\ldots + LB^{\sigma_1}_{s_1} B^{\sigma_2}_{s_2} \ldots\ldots + \ldots\ldots, \\ (s_1^{\sigma_1} s_2^{\sigma_2} \ldots\ldots)_\gamma = \ldots\ldots + L\,(p_1^{\pi_1} p_2^{\pi_2} \ldots\ldots)_\beta + \ldots\ldots \end{cases}$$

By direct multiplication of the product

$$C^{\pi_1}_{p_1} C^{\pi_2}_{p_2} \ldots\ldots,$$

we observe that L is composed of separations of the symmetric function

$$(s_1^{\sigma_1} s_2^{\sigma_2} \ldots\ldots)_\alpha,$$

of specification $(p_1^{\pi_1} p_2^{\pi_2} \ldots\ldots)$;

L is in fact composed of the tabular assemblages of separations which appear in the tables of the first two memoirs.

174. Hence in the result

$$(s_1^{\sigma_1} s_2^{\sigma_2} \ldots\ldots)_\gamma = \ldots\ldots + L\,(p_1^{\pi_1} p_2^{\pi_2} \ldots\ldots)_\beta + \ldots\ldots$$

L is composed of tabular separations of the symmetric function

$$(s_1^{\sigma_1} s_2^{\sigma_2} \ldots\ldots)_\alpha$$

of specification $(p_1^{\pi_1} p_2^{\pi_2} \ldots\ldots)$.

175. It is now easy to form any function

$$(s_1^{\sigma_1} s_2^{\sigma_2} \dots \dots)_\gamma .$$

Example I.

To form $(1^3 0)_\gamma$, first write down all the separations of $(1^3 0)$, and underneath them their respective specifications.

Separations:	$(1)^3(0)$,	$(10)(1)^2$,	$(1^2)(1)(0)$,	$(1^2)(10)$,	$(1^3)(0)$,	$(1^2 0)(1)$,	$(1^3 0)$,
Specifications:	$(1^3 0)$	(1^3)	(210)	(21)	(30)	(21)	(3).

We have now to form the C products corresponding to the several specifications and therein pick out the terms involving the B-product $B_1^3 B_0$.

$$\begin{aligned}
\text{Thus}\quad C_1^3 C_0 &= \dots\dots + (1)^3(0)\, B_1^3 B_0 &&+ \dots\dots,\\
C_1^3 &= \dots\dots + 3\,(1)^2(10)\, B_1^3 B_0 &&+ \dots\dots,\\
C_2 C_1 C_0 &= \dots\dots + (1^2)(1)(0)\, B_1^3 B_0 &&+ \dots\dots,\\
C_2 C_1 &= \dots\dots + \{(1^2)(10) + (1^2 0)(1)\}\, B_1^3 B_0 &&+ \dots\dots,\\
C_3 C_0 &= \dots\dots + (1^3)(0)\, B_1^3 B_0 &&+ \dots\dots,\\
C_3 &= \dots\dots + (1^3 0)\, B_1^3 B_0 &&+ \dots\dots
\end{aligned}$$

Hence

$$\begin{aligned}
(1^3 0)_\gamma = (1)^3(0)(1^3 0)_\beta + 3\,(1)^2(10)(1^3)_\beta + (1^2)(1)(0)(210)_\beta \\
+ \{(1^2)(10) + (1^2 0)(1)\}\,(21)_\beta + (1^3)(0)(30)_\beta + (1^3 0)(3)_\beta .
\end{aligned}$$

Observe that from this point of view $(1^3 0)_\gamma$ is a generating function for the tabular separations of symmetric function $(1^3 0)$.

176. If we now take the operator relation

$$\begin{array}{ll}
1 + {}_\beta G_0 + {}_\beta G_1 y \quad + \dots\dots = \displaystyle\prod_{s=1}^{s=n} \Bigg(1 + {}_\gamma G_0 + \alpha_s\, {}_\gamma G_1 y & + \dots\dots \\
\qquad + {}_\beta G_{-1} \dfrac{1}{y} + \dots\dots \qquad\qquad\qquad\qquad + \dfrac{1}{\alpha_s}\, {}_\gamma G_{-1} \dfrac{1}{y} & + \dots\dots \Bigg)
\end{array}$$

we obtain

$$\begin{aligned}
{}_\beta G_0 &= (0)\, {}_\gamma G_0 &&+ \dots\dots,\\
{}_\beta G_1 &= (1)\, {}_\gamma G_1 + (10)\, {}_\gamma G_1\, {}_\gamma G_0 &&+ \dots\dots,
\end{aligned}$$

where only those terms which are significant for present purposes have been retained.

Operating with these relations on opposite sides of the expression for $(1^3 0)_\gamma$, we have

$$(0)(1^3)_\gamma = (1)^3(0)(1^3)_\beta + (1^2)(1)(0)(21)_\beta + (1^3)(0)(3)_\beta ,$$

which is

$$(1^3)_\gamma = (1)^3(1^3)_\beta + (1^2)(1)(21)_\beta + (1^3)(3)_\beta$$

and

$$(1)(1^20)_\gamma + (10)(1^2)_\gamma = (1)\{(1)^2(0)(1^20)_\beta + 2\,(1)(10)(1^2)_\beta + (1^2)(0)(20)_\beta + (1^20)(2)_\beta\} \\ + (10)\{(1)^2(1^2)_\beta + (1^2)(2)_\beta\},$$

which is true, because the process explained gives

$$(1^20)_\gamma = (1)^2(0)(1^20)_\beta + 2\,(1)(10)(1^2)_\beta + (1^2)(0)(20)_\beta + (1^20)(2)_\beta$$

and
$$(1^2)_\gamma = (1)^2(1^2)_\beta + (1^2)(2)_\beta.$$

This example will serve to show the method of utilizing the operators for the derivation of new results from those already obtained.

177. The most important cases are those which have reference merely to partitions composed of parts which are positive, non-zero, integers.

We may write down a table of weight 4.

$$\begin{aligned}
(4)_\gamma &= (4)(4)_\beta,\\
(31)_\gamma &= (31)(4)_\beta + (3)(1)(31)_\beta,\\
(2^2)_\gamma &= (2^2)(4)_\beta + (2)^2(2^2)_\beta,\\
(21^2)_\gamma &= (21^2)(4)_\beta + (21)(1)(31)_\beta + 2\,(2)(1^2)(2^2)_\beta + (2)(1)^2(21^2)_\beta,\\
(1^4)_\gamma &= (1^4)(4)_\beta + (1^3)(1)(31)_\beta + (1^2)^2(2^2)_\beta + (1^2)(1)^2(21^2)_\beta + (1)^4(1^4)_\beta,
\end{aligned}$$

and thence we may immediately write down the corresponding operator relations, viz. writing in general

$$\frac{\overline{g_\lambda^l g_\mu^m \cdots\cdots}}{l!\ m! \cdots\cdots} = \overline{g_\lambda^{(l)} g_\mu^{(m)}} \cdots\cdots,$$

we have

$$\begin{aligned}
{}_\beta g_4 &= (4)\,{}_\gamma g_4,\\
\overline{{}_\beta g_3\,{}_\beta g_1} &= (31)\,{}_\gamma g_4 + (3)(1)\,\overline{{}_\gamma g_3\,{}_\gamma g_1},\\
\overline{{}_\beta g_2^{(2)}} &= (2^2)\,{}_\gamma g_4 + (2)^2\,\overline{{}_\gamma g_2^{(2)}},\\
\overline{{}_\beta g_2\,{}_\beta g_1^{(2)}} &= (21^2)\,{}_\gamma g_4 + (21)(1)\,\overline{{}_\gamma g_3\,{}_\gamma g_1} + 2\,(2)(1^2)\,\overline{{}_\gamma g_2^{(2)}} + (2)(1)^2\,\overline{{}_\gamma g_2\,{}_\gamma g_1^{(2)}},\\
\overline{{}_\beta g_1^{(4)}} &= (1^4)\,{}_\gamma g_4 + (1^3)(1)\,\overline{{}_\gamma g_3\,{}_\gamma g_1} + (1^2)^2\,\overline{{}_\gamma g_2^{(2)}} + (1^2)(1)^2\,\overline{{}_\gamma g_2\,{}_\gamma g_1^{(2)}} + (1)^4\,{}_\gamma g_1^{(4)},
\end{aligned}$$

and so in general it is clear that on the right-hand side we must obtain each tabular separation of the partition of the β operation on the left-hand side attached to a symbolic product of γ operations of the corresponding specification. These results may be utilized in a variety of ways, though at present I do not stop to further discuss them.

§13.

The Multiplication of Symmetric Functions.

178. Professor Cayley has in this Journal (Vol. VII, p. 2), laid down an algorithm for the multiplication of two symmetric functions which can be sym-

bolically expressed by partitions composed of positive integers. Moreover, I have, in the Messenger of Mathematics (Vol. XIV, p. 164), shown how to apply differential operators to the same purpose.

The method of operators may be used when the product contains any number whatever of functions, the coefficient of each term in the result being obtained by a separate and direct process.

Professor Cayley's process on the other hand, while more simple in the case of a product of two functions, is not adapted when the number of functions exceeds two; it may be employed when the partitions contain, as well, zero and negative integer parts.

Of this I give two examples.

179. *Example* I. To multiply

$$\Sigma\alpha^3\beta\gamma^0\delta^{-1} \text{ and } \Sigma\alpha^{-1}\beta^{-1};$$

that is, to form the product

$$(310\bar{1})(\bar{1}^2),$$

3	1	0	$\bar{1}$	$\bar{1}\bar{1}$	*F*	÷ 2	*M*	*C*
$\bar{1}$	$\bar{1}$				2	$(20^2\bar{1})$	2	2
$\bar{1}$		$\bar{1}$			2	$(21\bar{1}^2)$	2	2
$\bar{1}$			$\bar{1}$		2	$(210\bar{2})$	1	1
	$\bar{1}$	$\bar{1}$			2	$(30\bar{1}^2)$	2	2
	$\bar{1}$		$\bar{1}$		2	$(30^2\bar{2})$	2	2
		$\bar{1}$	$\bar{1}$		2	$(31\bar{1}\bar{2})$	1	1
$\bar{1}$				$\bar{1}$	2	$(210\bar{1}^2)$	2	2
	$\bar{1}$			$\bar{1}$	2	$(30^2\bar{1}^2)$	4	4
		$\bar{1}$		$\bar{1}$	2	$(31\bar{1}^3)$	6	6
			$\bar{1}$	$\bar{1}$	2	$(310\bar{1}\bar{2})$	1	1
				$\bar{1}\bar{1}$	1	$(310\bar{1}^3)$	6	3

where F means "frequency," M, multiplicity, and the coefficients in the column headed C are to be attached to the partitions in the same rows.

Reference should be made to Cayley (loc. cit.)

180. *Example* II. To multiply

$$\Sigma\alpha^0\beta^0\gamma^0 \text{ and } \Sigma\alpha^0\beta^0,$$

that is, to form the product

$$(0^3)(0^2),$$

we have

000	00	F	$\div 12$	M	C
00		6	(0^3)	6	3
0	0	6	(0^4)	24	12
	00	1	(0^5)	120	10

and the result is

$$(0^3)(0^2) = 3\,(0^3) + 12\,(0^4) + 10\,(0^5).$$

181. To establish the method of operators, suppose ϕ to be any rational integral function of

$$A_0,\ A_1,\ A_{-1},\ A_2,\ A_{-2},\ \ldots\ldots,$$

and let

$$\phi = \phi_1\phi_2\phi_3\ \ldots\ldots\ \phi_s;$$

if now we make the transformation of Art. 129, we shall have

$$\begin{aligned}
&\begin{pmatrix} 1 + G_0 + G^1\mu & + G_2\mu^2 & + \ldots\ldots \\ \quad + G_{-1}\dfrac{1}{\mu} & + G_{-2}\mu^2 & + \ldots\ldots \end{pmatrix}\phi \\
= &\begin{pmatrix} 1 + G_0 + G_1\mu & + G_2\mu^2 & + \ldots\ldots \\ \quad + G_{-1}\dfrac{1}{\mu} & + G_{-2}\dfrac{1}{\mu^2} & + \ldots\ldots \end{pmatrix}\phi_1 \\
\times &\begin{pmatrix} 1 + G_0 + G_1\mu & + G_2\mu^2 & + \ldots\ldots \\ \quad + G_{-1}\dfrac{1}{\mu} & + G_{-2}\dfrac{1}{\mu^2} & + \ldots\ldots \end{pmatrix}\phi_2 . \\
\times &\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\times &\begin{pmatrix} 1 + G_0 + G_1\mu & + G_2\mu^2 & + \ldots\ldots \\ \quad + G_{-1}\dfrac{1}{\mu} & + G_{-2}\dfrac{1}{\mu^2} & + \ldots\ldots \end{pmatrix}\phi_s
\end{aligned}$$

or

$$\left(\begin{array}{l} 1 + G_0 + G_1\mu \quad + G_2\mu^2 \quad + \dots \\ \qquad\quad + G_{-1}\frac{1}{\mu} + G_{-2}\frac{1}{\mu^2} + \dots \end{array}\right)\phi$$

$$= \prod_{s=1}^{s=s}\left(\begin{array}{l} \phi_s + G_0\phi_s + G_1\phi_s\mu \quad + G_2\phi_s\mu^2 \quad + \dots \\ \qquad\qquad + G_{-1}\phi_s\frac{1}{\mu} + G_{-2}\phi_s\frac{1}{\mu^2} + \dots \end{array}\right).$$

Whence, equating cofactors of like powers of μ, we obtain

$$G_\lambda\phi = \Sigma\Sigma\, G_{\lambda_1}\phi_{l_1} \,.\, G_{\lambda_2}\phi_{l_2} \,.\, G_{\lambda_3}\phi_{l_3} \dots,$$

where

(1). $(\lambda_1\lambda_2\lambda_3\dots)$ is any partition whatever of λ into positive, zero and negative integers.

(2). $\phi_{l_1}, \phi_{l_2}, \phi_{l_3}, \dots$ are different members of the set $\phi_1, \phi_2, \phi_3, \dots \phi_s$.

(3). The summation is taken for every partition of λ into positive, zero and negative parts.

(4). The summation is further taken for all the expressions obtained by permuting the numbers $l_1, l_2, l_3, \dots$ in all possible ways.

182. Hence if ϕ be a symmetric function expressed by a product of partitions, the operation G_λ is performed by abstracting every partition of λ in *all possible ways* from the product, one part at most being taken from each partition.

The process is in general simple enough and it must be made perfectly clear by a series of examples.

183. *Example* I. Let

$$\phi = (0^4)(0^3)(0^2).$$

Then

$$\begin{aligned} G_0\phi = {} & G_0(0^4).(0^3)(0^2) + (0^4).G_0(0^3).(0^2) + (0^4)(0^3).G_0(0^2) \\ & + G_0(0^4).G_0(0^3).(0^2) + G_0(0^4).(0^3).G_0(0^2) + (0^4).G_0(0^3).G_0(0^2) \\ & + G_0(0^4).G_0(0^3).G_0(0^2). \\ = {} & (0^3)(0^3)(0^2) + (0^4)(0^2)(0^2) + (0^4)(0^3)(0) \\ & + (0^3)(0^2)(0^2) + (0^3)(0^3)(0) + (0^4)(0^2)(0) \\ & + (0^3)(0^2)(0); \end{aligned}$$

where observe, in the first, second and third lines respectively, the partitions (0), (0^2), (0^3), each of which is a partition of zero, have been abstracted in all possible ways from the product in such wise that one part only is taken from each partition of the product.

184. *Example* II. Let

$$\phi = (0)^m;$$

then

$$G_0\phi = m\,(0)^{m-1} + \frac{1}{2!}\,m\,(m-1)(0)^{m-2} + \ldots. + \tfrac{1}{2}\,m\,(m-1)(0^2) + m\,(0) + 1,$$

the successive terms in the value of $G_0\phi$ corresponding to the abstraction of (0), (0^2), ... (0^{m-2}), (0^{m-1}), (0^m).

Thus $$G_0\,(0)^m = \{1 + (0)\}^m - (0)^m;$$

that is, $$G_0 n^m \;\; = (n+1)^m - n^m,$$

which is clearly right.

In general $$\begin{aligned} G_0 f(n) &= f(n+1) - f(n) \\ &= (E-1)\,f(n) \end{aligned}$$

in the notation of the calculus of finite differences. Hence G_0 is precisely equivalent to the symbol Δ of the same calculus, or

$$G_0 f(n) = \Delta f(n).$$

185. *Example* III. Let

$$\phi = (2)(1)(3\bar{1})(0).$$

To operate with G_1 we have to consider the following partitions of unity, viz:

$$(2\bar{1}),\ (2\bar{1}0),\ (1),\ (10).$$

Hence $$G_1\phi = (1)(3)(0) + (1)(3) + (2)(3\bar{1})(0) + (2)(3\bar{1}).$$

186. *Example* IV. Coming now to multiplication, suppose we are required to find the coefficient of the term

$$(30^2\bar{2})$$

in the product $$(310\bar{1})(\bar{1}^2).$$

If $$(310\bar{1})(\bar{1}^2) = \ldots. + A\,(30^2\bar{2}) + \ldots.,$$

then, operating throughout with $G_3G_0^2G_{-2}$, we find

$$G_3G_0^2G_{-2}\,(310\bar{1})(\bar{1}^2) = \ldots. + A + \ldots.,$$

where on the right-hand side the only term which is a simple number is the sought coefficient A; we have then to operate upon the product with the operation $G_3G_0^2G_{-2}$ and the numerical term that issues will be the result we seek.

To carry this out, we find

$$\begin{aligned} G_{-2}(310\bar{1})(\bar{1}^2) &= (310)(\bar{1}), \\ G_0 G_{-2}(310\bar{1})(\bar{1}^2) &= (31)(\bar{1}) + (30), \\ G_0^2 G_{-2}(310\bar{1})(\bar{1}^2) &= (3) + (3) = 2(3), \\ G_3 G_0^2 G_{-2}(310\bar{1})(\bar{1}^2) &= 2. \end{aligned}$$

Hence $(310\bar{1})(\bar{1}^2) = \ldots\ldots + 2(30^2\bar{2}) + \ldots\ldots,$

which agrees of course with the result obtained by Cayley's algorithm.

§14.

The Linear Partial Differential Operators of the Theory of Separations.

187. I propose to adapt the operations

$$g_0,\ g_1,\ g_{-1},\ g_2,\ g_{-2},\ldots\ldots,$$

so that they may be performed on the expression of any symmetric function in terms of separations of a given partition.

It will be remembered that a partition is separated into partitions termed "separates." Any combination whatever of the parts of a partition may present itself as a separate. If the separable partition be

$$(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3}\ldots\ldots),$$

precisely $(l_1+1)(l_2+1)(l_3+1)\ldots\ldots - 1$ distinct separates *may* occur.

It is necessary to consider all these separates as independent variables.

Let then $(\ldots\ldots 2^{p_2+\pi_2}1^{p_1+\pi_1}0^{p_0+\pi_0}\bar{1}^{p_{-1}+\pi_{-1}}\bar{2}^{p_{-2}+\pi_{-2}}\ldots\ldots)$

be any separate of a given separable partition

$$P,$$

then, by a known theorem,

$$g_s \equiv \Sigma g_s(\ldots 2^{p_2+\pi_2}1^{p_1+\pi_1}0^{p_0+\pi_0}\bar{1}^{p_{-1}+\pi_{-1}}\bar{2}^{p_{-2}+\pi_{-2}}\ldots)\,\partial_{(\ldots 2^{p_2+\pi_2}1^{p_1+\pi_1}0^{p_0+\pi_0}\bar{1}^{p_{-1}+\pi_{-1}}\bar{2}^{p_{-2}+\pi_{-2}}\ldots)},$$

the summation being in regard to every separate. Moreover, we have proved the relation

$$\frac{(-)^s}{s}g_s = \sum\frac{(-)^{\Sigma\pi}(\Sigma\pi-1)!}{\ldots\ldots\pi_2!\,\pi_1!\,\pi_0!\,\pi_{-1}!\,\pi_{-2}!\ldots\ldots}\ldots\ldots G_2^{\pi_2}G_1^{\pi_1}G_0^{\pi_0}G_{-1}^{\pi_{-1}}G_{-2}^{\pi_{-2}}\ldots\ldots,$$

when the summation is in regard to the solutions in positive, zero and negative integers of the indeterminate equation

$$\Sigma t\pi_t = s,$$

and also

$$\ldots\ldots G_2^{\pi_2} G_1^{\pi_1} G_0^{\pi_0} G_{-1}^{\pi_{-1}} G_{-2}^{\pi_{-2}} \ldots\ldots (\ldots\ldots 2^{p_2+\pi_2} 1^{p_1+\pi_1} 0^{p_0+\pi_0} \bar{1}^{p_{-1}+\pi_{-1}} \bar{2}^{p_{-2}+\pi_{-2}} \ldots\ldots)$$
$$= (\ldots\ldots 2^{p_2} 1^{p_1} 0^{p_0} \bar{1}^{p_{-1}} \bar{2}^{p_{-2}} \ldots\ldots).$$

188. Hence $$\frac{(-)^s}{s} g_s$$

$$\equiv \sum\sum \frac{(-)^{\Sigma\pi}(\Sigma\pi - 1)!}{\ldots\ldots \pi_2!\, \pi_1!\, \pi_0!\, \pi_{-1}!\, \pi_{-2}! \ldots\ldots}$$
$$(\ldots\ldots 2^{p_2} 1^{p_1} 0^{p_0} \bar{1}^{p_{-1}} \bar{2}^{p_{-2}} \ldots\ldots)\, \partial_{(\ldots\ldots 2^{p_2+\pi_2} 1^{p_1+\pi_1} 0^{p_0+\pi_0} \bar{1}^{p_{-1}+\pi_{-1}} \bar{2}^{p_{-2}+\pi_{-2}} \ldots\ldots)},$$

the summation being in regard

(1) to every separate;

(2) to every solution of the indeterminate equation $\Sigma t\pi_t = s$ in positive, zero, and negative integers.

189. To take a concrete case, consider the separable partition

$$(21^{n-2});$$

we then have

$$g_1 \equiv \partial_{(1)} + (1)\,\partial_{(1^2)} + (1^2)\,\partial_{(1^3)} + \ldots\ldots + (1^{n-3})\,\partial_{(1^{n-2})}$$
$$+ (2)\,\partial_{(21)} + (21)\,\partial_{(21^2)} + (21^2)\,\partial_{(21^3)} + \ldots\ldots + (21^{n-3})\,\partial_{(21^{n-2})},$$
$$g_s \equiv \partial_{(1^s)} + (1)\,\partial_{(1^{s+1})} + (1^2)\,\partial_{(1^{s+2})} + \ldots\ldots + (1^{n-s-2})\,\partial_{(1^{n-2})}$$
$$+ (2)\,\partial_{(21^s)} + (21)\,\partial_{(21^{s+1})} + \ldots\ldots + (21^{n-s-2})\,\partial_{(21^{n-2})}$$
$$- s\,\{\partial_{(21^{s-2})} + (1)\,\partial_{(21^{s-1})} + \ldots\ldots + (1^{n-s})\,\partial_{(21^{n-2})}\},$$

relations which are obtained at once from the general formula.

190. Again, consider the separable partition

$$(321^2);$$

the separates are

$$\begin{array}{llll} (1) & , (2) & , (3) & , \\ (1^2) & , (21) & , (31) & , (32), \\ (21^2) & , (31^2), & (321), & \\ (321^2); & & & \end{array}$$

then

$$g_1 \equiv \partial_{(1)} + (1)\,\partial_{(1^2)} + (2)\,\partial_{(21)} + (21)\,\partial_{(21^2)} + (3)\,\partial_{(31)} + (31)\,\partial_{(31^2)} + (32)\,\partial_{(321)} + (321)\partial_{(321^2)};$$

wherein only those separates which contain the part unity occur as independent variables. Also

$$g_2 \equiv \partial_{(1^2)} + (2)\,\partial_{(21^2)} + (3)\,\partial_{(31^2)} + (32)\,\partial_{(321^2)}$$
$$- 2\,\{\partial_{(2)} + (1)\,\partial_{(21)} + (1^2)\,\partial_{(21^2)} + (3)\,\partial_{(32)} + (31)\,\partial_{(321)} + (31^2)\,\partial_{(321^2)}\},$$

wherein the operation is comprised of two portions corresponding to the partitions (1^2) and (2) of the number 2.

In the first and second portions respectively the independent variables contain the partitions (1^2) and (2).

This should be compared with the known formula

$$s_2 = a_1^2 - 2a_2.$$

Similarly

$$\begin{aligned} g_3 \equiv &- 3\{\partial_{(21)} + (1)\,\partial_{(21^2)} + (3)\,\partial_{(321)} + (31)\,\partial_{(321^2)}\} \\ &+ 3\{\partial_{(3)} + (1)\,\partial_{(31)} + (1^2)\,\partial_{(31^2)} + (2)\,\partial_{(32)} + (21)\,\partial_{(321)} + (21^2)\,\partial_{(321^2)}\} \end{aligned}$$

in correspondence with

$$s_3 = \ldots\ldots - 3a_2a_1 + 3a_3,$$

and

$$\begin{aligned} g_4 \equiv &- 4\partial_{(21^2)} \\ &+ 4\{\partial_{(31)} + (1)\,\partial_{(31^2)} + (2)\,\partial_{(321)} + (21)\,\partial_{(321^2)}\} \end{aligned}$$

in correspondence with

$$s_4 = -4a_2a_1^2 + 4a_3a_1 + \ldots\ldots$$

Also

$$\begin{aligned} g_5 &\equiv 5\{\partial_{(31^2)} + (2)\,\partial_{(321^2)}\} - 5\{\partial_{(32)} + (1)\,\partial_{(321)} + (1^2)\,\partial_{(321^2)}\}, \\ g_6 &\equiv -12\{\partial_{(321)} + (1)\,\partial_{(321^2)}\}, \\ g_7 &\equiv -21\partial_{(321^2)}, \end{aligned}$$

in correspondence respectively with

$$\begin{aligned} s_5 &= 5a_3a_1^2 + \ldots\ldots, \\ s_6 &= -12a_3a_2a_1 + \ldots\ldots, \\ s_7 &= -21a_3a_2a_1^2 + \ldots\ldots \end{aligned}$$

Any of these results may be easily verified. Thus from the first memoir

$$(7) = \ldots\ldots - \tfrac{1}{3}(321^2) + \ldots\ldots,$$

and since

$$g_7(7) = +7,$$

it is obvious that g_7 and $-21\partial_{(321^2)}$ are equivalent operations.

191. The formation of these operators is seen to be particularly simple in the case of a separable partition composed merely of positive (non-zero) integers. They are written down at once from the expression of Vandermonde for the sums of the powers of the quantities in terms of the elementary (that is, unitary) symmetric functions.

Let us now consider a separable partition which contains as well zero and negative integers.

For example, the partition

$$(210^2\bar{1})$$

of weight 2.

We must now consider the expressions of the sums of the powers in terms of the arguments

$$A_0,\ A_1,\ A_{-1},\ A_2,\ A_{-2},\ \ldots\ldots,$$

merely retaining necessary terms of the infinite expressions

$$\begin{aligned}
s_{-1} &= \quad A_{-1} - A_0A_{-1} + A_0^2A_{-1} + \ldots\ldots, \\
s_0 &= \quad A_0 - \tfrac{1}{2}A_0^2 - A_1A_{-1} + 2A_1A_0A_{-1} - 3A_1A_0^2A_{-1} + \ldots\ldots, \\
s_1 &= \quad A_1 - A_1A_0 + A_1A_0^2 - A_2A_{-1} + 2A_2A_0A_{-1} - 3A_2A_0^2A_{-1} + \ldots\ldots, \\
s_2 &= -2A_2 + 2A_2A_0 - 2A_2A_0^2 - 4A_2A_1A_{-1} + 12A_2A_1A_0A_{-1} - 24A_2A_1A_0^2A_{-1} + \ldots, \\
s_3 &= -3A_2A_1 + 6A_2A_1A_0 - 9A_2A_1A_0^2 + \ldots\ldots
\end{aligned}$$

192. Hence the following expressions for

$$g_{-1},\ g_0,\ g_1,\ g_2,\ g_3,$$

viz.

$$\begin{aligned}
g_{-1} \equiv \{\partial_{(\bar{1})} &+ (0)\,\partial_{(0\bar{1})} + (1)\,\partial_{(1\bar{1})} + (2)\,\partial_{(2\bar{1})} + (0^2)\,\partial_{(0^2\bar{1})} + (10)\,\partial_{(10\bar{1})} + (20)\,\partial_{(20\bar{1})} \\
&+ (21)\,\partial_{(21\bar{1})} + (210)\,\partial_{(210\bar{1})} + (20^2)\,\partial_{(20^2\bar{1})} + (10^2)\,\partial_{(10^2\bar{1})} + (210^2)\,\partial_{(210^2\bar{1})}\} \\
-\{\partial_{(0\bar{1})} &+ (0)\,\partial_{(0^2\bar{1})} + (1)\,\partial_{(10\bar{1})} + (2)\,\partial_{(20\bar{1})} + (10)\,\partial_{(10^2\bar{1})} + (20)\,\partial_{(20^2\bar{1})} \\
&+ (21)\,\partial_{(210\bar{1})} + (210)\,\partial_{(210^2\bar{1})}\} \\
+\{\partial_{(0^2\bar{1})} &+ (1)\,\partial_{(10^2\bar{1})} + (2)\,\partial_{(20^2\bar{1})} + (21)\,\partial_{(210^2\bar{1})}\};
\end{aligned}$$

$$\begin{aligned}
g_0 \equiv \partial_{(0)} &+ (2)\,\partial_{(20)} + (1)\,\partial_{(10)} + (0)\,\partial_{(0^2)} + (\bar{1})\,\partial_{(0\bar{1})} + (21)\,\partial_{(210)} \\
&+ (0\bar{1})\,\partial_{(0^2\bar{1})} + (10)\,\partial_{(10^2)} + (20)\,\partial_{(20^2)} + (1\bar{1})\,\partial_{(10\bar{1})} + (2\bar{1})\,\partial_{(20\bar{1})} \\
&+ (210)\,\partial_{(210^2)} + (21\bar{1})\,\partial_{(210\bar{1})} + (20\bar{1})\,\partial_{(20^2\bar{1})} + (10\bar{1})\,\partial_{(10^2\bar{1})} + (210\bar{1})\,\partial_{(210^2\bar{1})} \\
-\tfrac{1}{2}\{\partial_{(0^2)} &+ (1)\,\partial_{(10^2)} + (2)\,\partial_{(20^2)} + (\bar{1})\,\partial_{(0^2\bar{1})} + (21)\,\partial_{(210^2)} \\
&+ (2\bar{1})\,\partial_{(20^2\bar{1})} + (1\bar{1})\,\partial_{(10^2\bar{1})} + (21\bar{1})\,\partial_{(210^2\bar{1})}\} \\
-\ \{\partial_{(1\bar{1})} &+ (0)\,\partial_{(10\bar{1})} + (0^2)\,\partial_{(10^2\bar{1})} + (2)\,\partial_{(21\bar{1})} + (20)\,\partial_{(210\bar{1})} + (20^2)\,\partial_{(210^2\bar{1})}\} \\
+2\{\partial_{(10\bar{1})} &+ (0)\,\partial_{(10^2\bar{1})} + (2)\,\partial_{(210\bar{1})} + (20)\,\partial_{(210^2\bar{1})}\} \\
-3\{\partial_{(10^2\bar{1})} &+ (2)\,\partial_{(210^2\bar{1})}\};
\end{aligned}$$

$$\begin{aligned}
g_1 \equiv \partial_{(1)} &+ (0)\,\partial_{(10)} + (\bar{1})\,\partial_{(1\bar{1})} + (2)\,\partial_{(21)} + (0^2)\,\partial_{(0^21)} + (0\bar{1})\,\partial_{(10\bar{1})} + (2\bar{1})\,\partial_{(21\bar{1})} \\
&+ (20)\,\partial_{(210)} + (0^2\bar{1})\,\partial_{(10^2\bar{1})} + (20\bar{1})\,\partial_{(210\bar{1})} + (20^2)\,\partial_{(210^2)} + (20^2\bar{1})\,\partial_{(210^2\bar{1})} \\
-\{\partial_{(10)} &+ (\bar{1})\,\partial_{(10\bar{1})} + (0)\,\partial_{(10^2)} + (2)\,\partial_{(210)} + (0\bar{1})\,\partial_{(10^2\bar{1})} + (2\bar{1})\,\partial_{(210\bar{1})} \\
&+ (20)\,\partial_{(210^2)} + (20\bar{1})\,\partial_{(210^2\bar{1})}\} \\
+\{\partial_{(10^2)} &+ (\bar{1})\,\partial_{(10^2\bar{1})} + (2)\,\partial_{(210^2)} + (2\bar{1})\,\partial_{(210^2\bar{1})}\} \\
-\{\partial_{(2\bar{1})} &+ (0)\,\partial_{(20\bar{1})} + (1)\,\partial_{(21\bar{1})} + (0^2)\,\partial_{(20^2\bar{1})} + (10)\,\partial_{(210\bar{1})} + (10^2)\,\partial_{(210^2\bar{1})}\} \\
+2\{\partial_{(20\bar{1})} &+ (0)\,\partial_{(20^2\bar{1})} + (1)\,\partial_{(210\bar{1})} + (10)\,\partial_{(210^2\bar{1})}\} \\
-3\{\partial_{(20^2\bar{1})} &+ (1)\,\partial_{(210^2\bar{1})}\};
\end{aligned}$$

$$\begin{aligned}
g_2 \equiv & -2\{\partial_{(2)}+(\bar{1})\,\partial_{(2\bar{1})}+(0)\,\partial_{(20)}+(1)\,\partial_{(21)}+(0\bar{1})\,\partial_{(20\bar{1})}+(0^2)\,\partial_{(20^2)} \\
& \qquad +(1\bar{1})\,\partial_{(21\bar{1})}+(10)\,\partial_{(210)}+(0^2\bar{1})\,\partial_{(20^2\bar{1})}+(10\bar{1})\,\partial_{(210\bar{1})} \\
& \qquad +(10^2)\,\partial_{(210^2)}+(10^2\bar{1})\,\partial_{(210^2\bar{1})}\} \\
& +2\{\partial_{(20)}+(\bar{1})\,\partial_{(20\bar{1})}+(0)\,\partial_{(20^2)}+(1)\,\partial_{(210)}+(0\bar{1})\,\partial_{(20^2\bar{1})} \\
& \qquad +(1\bar{1})\,\partial_{(210\bar{1})}+(10)\,\partial_{(210^2)}+(10\bar{1})\,\partial_{(210^2\bar{1})}\} \\
& -2\{\partial_{(20^2)}+(\bar{1})\,\partial_{(20^2\bar{1})}+(1)\,\partial_{(210^2)}+(1\bar{1})\,\partial_{(210^2\bar{1})}\} \\
& -4\{\partial_{(21\bar{1})}+(0)\,\partial_{(210\bar{1})}+(0^2)\,\partial_{(210^2\bar{1})}\} \\
& +12\{\partial_{(210\bar{1})}+(0)\,\partial_{(210^2\bar{1})}\} \\
& -24\partial_{(210^2\bar{1})}, \\
g_3 \equiv & -3\{\partial_{(21)}+(\bar{1})\,\partial_{(21\bar{1})}+(0)\,\partial_{(210)}+(0\bar{1})\,\partial_{(210\bar{1})}+(0^2)\,\partial_{(210^2)}+(0^2\bar{1})\,\partial_{(210^2\bar{1})}\} \\
& +6\{\partial_{(210)}+(\bar{1})\,\partial_{(210\bar{1})}+(0)\,\partial_{(210^2)}+(0\bar{1})\,\partial_{(210^2\bar{1})}\} \\
& -9\{\partial_{(210^2)}+(\bar{1})\,\partial_{(210^2\bar{1})}\}.
\end{aligned}$$

193. These operations appear to be somewhat complicated, but it will be seen subsequently that for practical use they may in general be broken up into effective fragments. Their application to calculations in the separation theory must be reserved for a future occasion; I hope then also to bring forward a new theorem of distribution of an extended character and to apply the method of this memoir to its analytical solution.

The main result was communicated by me verbally to the London Mathematical Society at its February meeting.

Woolwich, England, *April* 24, 1889.